A Cartan type identity for isoparametric hypersurfaces in symmetric spaces

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Abstract

In this paper, we obtain a Cartan type identity for curvature-adapted isoparametric hypersurfaces in symmetric spaces of compact type or non-compact type. This identity is a generalization of Cartan-D'Atri's identity for curvature-adapted(=amenable) isoparametric hypersurfaces in rank one symmetric spaces. In the case where the ambient symmetric space is of compact type, the proof is performed by showing the minimality of a focal submanifold of the hypersurface, where we note that the minimality is shown by investigating the lift of the focal submanifold to a Hilbert space through a Riemannian submersion in the case where the rank of the symmetric space is greater than one. In the case where the ambient symmetric space is of non-compact type, the proof is performed by showing the minimality of a focal submanifold of the complexification of the hypersurface, where we note that the minimality is shown by investigating the lift of the focal submanifold to an infinite dimensional anti-Kaehlerian space through an anti-Kaehlerian submersion in the case where the rank of the symmetric space is greater than one.

Keywords; isoparametric hypersurface, principal curvature, focal radius, complex focal radius

1 Introduction

An isoparametric hypersurface in a general Riemannian manifold is a complete connected hypersurface whose sufficiently close parallel hypersurfaces are of constant mean curvature (see [HLO] for example). It is known that all isoparametric hypersurfaces in a symmetric space of compact type are equifocal in the sense of [TT] and that, conversely all equifocal hypersurfaces are isoparametric (see [HLO]). Also, it is known that all isoparametric hypersurfaces in a symmetric space of non-compact type are complex equifocal in the sense of [Koi2] and that, conversely, all curvature-adapted complex equifocal hypersurfaces are isoparametric (see Theorem 15 of [Koi3]), where the curvature-adaptedness implies that, for a unit normal vector v, the (normal) Jacobi operator $R(\cdot, v)v$ preserves the tangent space invariantly and commutes with the shape operator A for v, where R is the curvature tensor of the ambient space. It is known that principal orbits of a Hermann action (i.e., the action of a symmetric subgroup of G) of cohomogeneity one on a symmetric space G/K of compact type are curvature-adapted and equifocal (see ([GT]). Hence they are isoparametric hypersurfaces. On the other hand, we [Koi4,7] showed that the principal orbits of a Hermann type action (i.e., the action of a (not necessarily compact) symmetric subgroup of G) of cohomogeneity one on a symmetric space G/K of non-compact type are

curvature-adapted and complex equifocal, and they have no focal point of non-Euclidean type on the ideal boundary of G/K. Hence they are isoparametric hypersurfaces.

For an isoparametric hypersurface M in a real space form N of constant curvature c, it is known that the following Cartan's identity holds:

$$\sum_{\lambda\in\mathrm{Spec}A\setminus\{\lambda_0\}}\frac{c+\lambda\lambda_0}{\lambda-\lambda_0}\times m_\lambda=0 \tag{1.1}$$

for any $\lambda_0\in\mathrm{Spec}A$, where A is the shape operator of M and $\mathrm{Spec}A$ is the spectrum of A, m_λ is the multiplicity of λ. Here we note that all hypersurfaces in a real space form are curvature-adapted. In general cases, this identity is shown in algebraic method. Also, It is shown in geometrical method in the following three cases:

(i) $c=0,\ \lambda_0\neq 0$,
(ii) $c>0,\ \lambda_0$: any eigenvalue of A_v,
(iii) $c<0,\ |\lambda_0|>\sqrt{-c}$.

In detail, it is shown by showing the minimality of the focal submanifold for λ_0 and using this fact.

Let $H\curvearrowright G/K$ be a cohomogeneity one action of a compact group $H\,(\subset G)$ on a rank one symmetric space G/K and M a principal orbit of this action. Since the H-action is of cohomogeneity one, it is hyperpolar. Hence M is an equifocal (hence isoparametric) hypersurface (see [HPTT]). In 1979, J. E. D'Atri [D] obtained a Cartan type identity for M in the case where M is amenable (i.e., curvature-adapted). On the other hand, in 1989-1991, J. Berndt [B1,2] obtained a Cartan type identity (in algebraic method) for curvature-adapted hypersurfaces with constant principal curvature in rank one symmetric spaces other than spheres and hyperbolic spaces. Here we note that, for a curvature-adapted hypersurface in a rank one symmetric space of non-compact type, it has constant principal curvature if and only if it is isoparametric.

Let M be a hypersurface in a symmetric space $N=G/K$ of compact type or non-compact type and v a unit normal vector field of M. Set $R(v_x):=R(\cdot,v_x)v_x|_{T_xM}$, where R is the curvature tensor of N. For each $r\in\mathbb{R}$, we define a function τ_r over $[0,\infty)$ by

$$\tau_r(s):=\begin{cases}\dfrac{\sqrt{s}}{\tan(r\sqrt{s})} & (s>0)\\ \dfrac{1}{r} & (s=0)\end{cases}$$

Also, for each $r\in\mathbb{C}$, we define a complex-valued function $\hat{\tau}_r$ over $(-\infty,0]$ by

$$\hat{\tau}_r(s):=\begin{cases}\dfrac{\mathbf{i}\sqrt{-s}}{\tan(\mathbf{i}r\sqrt{-s})} & (s<0)\\ \dfrac{1}{r} & (s=0),\end{cases}$$

where $\mathbf{i}$ is the imaginary unit. In this paper, we first prove the following Cartan type identity for a curvature-adapted isoparametric hypersurface in a simply connected symmetric space of compact type.

Theorem A. *Let M be a curvature-adapted isoparametric hypersurface in a simply connected symmetric space $N := G/K$ of compact type. For each focal radius r_0 of M, we have*

$$\sum_{(\lambda,\mu)\in S^x_{r_0}} \frac{\mu+\lambda\tau_{r_0}(\mu)}{\lambda-\tau_{r_0}(\mu)} \times m_{\lambda,\mu} = 0, \tag{1.2}$$

where $S^x_{r_0} := \{(\lambda,\mu) \in \mathrm{Spec}A_x \times \mathrm{Spec}R(v_x) \,|\, \mathrm{Ker}(A_x - \lambda I) \cap \mathrm{Ker}(R(v_x) - \mu I) \neq \{0\},\ \lambda \neq \tau_{r_0}(\mu)\}$ and $m_{\lambda,\mu} := \dim(\mathrm{Ker}(A_x - \lambda I) \cap \mathrm{Ker}(R(v_x) - \mu I))$.

Remark 1.1. (i) If $\mathrm{Ker}(A_x - \lambda_0 I) \cap \mathrm{Ker}(R(v_x) - \mu_0 I)$ is included by the focal space for the focal radius r_0, then we have $\tau_{r_0}(\mu_0) = \lambda_0$.

(ii) If G/K is a sphere of constant curvature c, then $\mathrm{Spec}R(v_x) = \{c\}$ and $\tau_{r_0}(c)$ is equal to the principal curvature corresponding to r_0. Hence the identity (1.2) coincides with (1.1).

(iii) In the case where G/K is a rank one symmetric space of compact type, the identity (1.2) coincides with the identity obtained by J. E. D'Atri [D] (see Theorems 3.7 and 3.9 of [D]).

(iv) In the case where G/K is a rank one symmetric space of compact type other than spheres, the identity (1.2) is different from the identity obtained by J. Berndt [B1,2].

The proof of Theorem A is performed by showing **the minimality of the focal submanifold** $F := \{\exp^\perp(r_0 v_x) \,|\, x \in M\}$ of M, where $\exp^\perp$ is the normal exponential map of M. In the case where G/K is of rank greater than one and M is not homogeneous, the proof of the minimality of F is performed by showing **the minimality of the lifted submanifold** $\widetilde{F} := (\pi \circ \phi)^{-1}(F)$ of F to the path space $H^0([0,1], \mathfrak{g}^{\mathbf{c}})$, where ϕ is the parallel transport map for G (which is a Riemannian submersion o $H^0([0,1], \mathfrak{g})$ onto G and π is the natural projection of G onto G/K (which also is a Riemannian submersion).

Next, in this paper, we prove the following Cartan type identity for a curvature-adapted isoparametric C^ω-hypersurface in a symmetric space of non-compact type, where C^ω means the real analyticity.

Theorem B. *Let M be a curvature-adapted isoparametric C^ω-hypersurface in a symmetric space $N := G/K$ of non-compact type. Assume that M has no focal point of non-Euclidean type on the ideal boundary $N(\infty)$ of N. Then M admits a complex focal radius and , for each complex focal radius r_0 of M, we have*

$$\sum_{(\lambda,\mu)\in S^x_{r_0}} \frac{\mu+\lambda\hat{\tau}_{r_0}(\mu)}{\lambda-\hat{\tau}_{r_0}(\mu)} \times m_{\lambda,\mu} = 0, \tag{1.3}$$

where $S^x_{r_0} := \{(\lambda,\mu) \in \mathrm{Spec}A_x \times \mathrm{Spec}R(v_x) \,|\, \mathrm{Ker}(A_x - \lambda I) \cap \mathrm{Ker}(R(v_x) - \mu I) \neq \{0\},\ \lambda \neq \hat{\tau}_{r_0}(\mu)\}$ and $m_{\lambda,\mu} := \dim(\mathrm{Ker}(A_x - \lambda I) \cap \mathrm{Ker}(R(v_x) - \mu I))$.

Remark 1.2. (i) The notion of a complex focal radius was introduced In [Koi2]. This quantity indicates the position of a focal point of the complexification $M^{\mathbf{c}}$ ($\subset G^{\mathbf{c}}/K^{\mathbf{c}}$) of a submanifold M in a symmetric space G/K of non-compact type (see [Koi3]).

(ii) If $\mathrm{Ker}(A_x - \lambda_0 I) \cap \mathrm{Ker}(R(v_x) - \mu_0 I)$ is included by the focal space for the complex focal radius r_0, then we have $\hat{\tau}_{r_0}(\mu_0) = \lambda_0$.

(iii) If G/K is a hyperbolic space of constant curvature c, then $\mathrm{Spec}R(v_x) = \{c\}$ and $\hat{\tau}_{r_0}(c)$ is equal to the principal curvature corresponding to r_0. Hence the identity (1.3) coincides with (1.1).

(iv) In the case where G/K is a rank one symmetric space of non-compact type and r_0 is a real focal radius, the identity (1.3) coincides with the identity obtained by J. E. D'Atri [D] (see Theorems 3.7 and 3.9 of [D]).

(v) In the case where G/K is a rank one symmetric space of non-compact type other than hyperbolic spaces, the identity (1.3) is different from the identity obtained by J. Berndt [B1,2].

(vi) For a curvature-adapted and isoparametric hypersurface M in G/K, the following conditions (a) $\sim$ (c) are equivalent:

(a) M has no focal point of non-Euclidean type on $N(\infty)$,

(b) M is proper complex equifocal in the sense of [Koi4],

(c) $\mathrm{Ker}(A_x \pm \sqrt{-\mu}I) \cap \mathrm{Ker}(R(v_x) - \mu I) = \{0\}$ holds for each $\mu \in \mathrm{Spec}R(v_x) \setminus \{0\}$.

(vii) Principal orbits of a Hermann type action of cohomogeneity one on G/K are curvature-adapted isoparametric C^ω-hypersurface having no focal point of non-Euclidean type on $N(\infty)$ (see Theorem B of [Koi4] and the above (iii)).

The proof of Theorem B is performed by showing **the minimality of the focal submanifold** $F := \{\exp^\perp((\mathrm{Re}\, r_0)v_x + (\mathrm{Im}\, r_0)Jv_x) \,|\, x \in M^{\mathbf{c}}\}$ of the complexification $M^{\mathbf{c}}$ of M (see Fig.1), where $\exp^\perp$ is the normal exponential map of the submanifold $M^{\mathbf{c}}$ in $G^{\mathbf{c}}/K^{\mathbf{c}}$, J is the complex structure of $G^{\mathbf{c}}/K^{\mathbf{c}}$ and v is a unit normal vector field of M (in G/K). Here we note that $\exp^\perp((\mathrm{Re}\, r_0)v_x + (\mathrm{Im}\, r_0)Jv_x)$ is equal to the point $\gamma^{\mathbf{c}}_{v_x}(r_0)$ of the complexified geodesic $\gamma^{\mathbf{c}}_{v_x}$ in $G^{\mathbf{c}}/K^{\mathbf{c}}$. In the case where G/K is of rank greater than one and M is not homogeneous, the proof of the minimality of F is performed by showing **the minimality of the lift** $\widetilde{F} := (\pi \circ \phi)^{-1}(F)$ **of** F **to the path space** $H^0([0,1], \mathfrak{g}^{\mathbf{c}})$, where ϕ is the parallel transport map for $G^{\mathbf{c}}$ (which is an anti-Kaehlerian submersion o $H^0([0,1], \mathfrak{g}^{\mathbf{c}})$ onto $G^{\mathbf{c}}$) and π is the natural projection of $G^{\mathbf{c}}$ onto $G^{\mathbf{c}}/K^{\mathbf{c}}$ (which also is an anti-Kaehlerian submersion). Here we note that the minimality of F is trivial in the case where M is homogeneous. By using Theorem B, we prove the following fact for the number of distinct principal curvatures of a curvature-adapted isoparametric C^ω-hypersurfaces in a symmetric sapce of non-compact type.

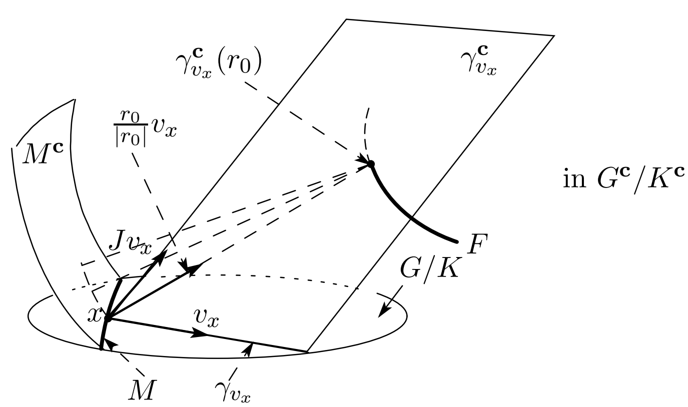


Fig. 1.

Theorem C. *Let M be a curvature-adapted isoparametric C^ω-hypersurface in a symmetric space $N := G/K$ of non-compact type. Assume that M has no focal point of non-Euclidean type on $N(\infty)$. Then the following statements* (i) *and* (ii) *hold:*

(i) $\sharp\mathrm{Spec}\left(A_x|_{\mathrm{Ker}(R(v_x)-\mu I)}\right) \leq 2$ *for* $\mu \in \mathrm{Spec}R(v_x)$, *where x is an arbitrary point of M and $\sharp(\cdot)$ implies the cardinal number of* $(\cdot)$.

$$\text{(ii) } \sharp\mathrm{Spec}\,A_x \leq \begin{cases} \sharp(\triangle_+|_{\mathbf{R}g_*^{-1}v_x} \setminus (\triangle_+|_{\mathbf{R}g_*^{-1}v_x})^1) \times 2 + \sharp(\triangle_+|_{\mathbf{R}g_*^{-1}v_x})^1 + 2 \\ \hfill (\mathrm{rank}(G/K) \geq 3) \\ \sharp(\triangle_+|_{\mathbf{R}g_*^{-1}v_x} \setminus (\triangle_+|_{\mathbf{R}g_*^{-1}v_x})^1) \times 2 + \sharp(\triangle_+|_{\mathbf{R}g_*^{-1}v_x})^1 + 1 \\ \hfill (\mathrm{rank}(G/K) = 2) \\ \sharp(\triangle_+|_{\mathbf{R}g_*^{-1}v_x} \setminus (\triangle_+|_{\mathbf{R}g_*^{-1}v_x})^1) \times 2 + \sharp(\triangle_+|_{\mathbf{R}g_*^{-1}v_x})^1 \\ \hfill (\mathrm{rank}(G/K) = 1). \end{cases}$$

Here $\triangle_+|_{\mathbf{R}g_*^{-1}v_x} := \{\alpha|_{\mathbf{R}g_*^{-1}v_x} \,|\, \alpha \in \triangle_+\}$, *where $\triangle_+$ is the positive root system (under a lexicographic ordering of $\mathfrak{a}^*$) of the root system $\triangle$ of G/K with respect to a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p} = T_{eK}(G/K)$ contaning $g_*^{-1}v_x$ ($x = gK$) and $\mathbb{R}g_*^{-1}v_x$ is the one dimensional subspace of $\mathfrak{a}$ generated by $g_*^{-1}v_x$, and* $(\triangle_+|_{\mathbf{R}g_*^{-1}v_x})^1 := \{\beta \in \triangle_+|_{\mathbf{R}g_*^{-1}v_x} \,|\, \dim(\sum_{\alpha \in \triangle_+ \text{ s.t. } \alpha|_{\mathbf{R}g_*^{-1}v_x} = \beta} \mathfrak{p}_\alpha) = 1\}$ ($\mathfrak{p}_\alpha$ *: the root space for* α).

Remark 1.3. (i) Clearly we have

$$\sharp(\triangle_+|_{\mathbf{R}g_*^{-1}v_x} \setminus (\triangle_+|_{\mathbf{R}g_*^{-1}v_x})^1) \times 2 + \sharp(\triangle_+|_{\mathbf{R}g_*^{-1}v_x})^1 \leq \sharp(\triangle_+ \setminus \triangle_+^1) + \sharp\triangle_+^1,$$

where $\triangle_+^1 := \{\alpha \in \triangle_+ \,|\, \dim \mathfrak{p}_\alpha = 1\}$. Hence we have

$$\sharp\mathrm{Spec}\,A_x \leq \begin{cases} \sharp(\triangle_+ \setminus \triangle_+^1) \times 2 + \sharp\triangle_+^1 + 2 & (\mathrm{rank}(G/K) \geq 3) \\ \sharp(\triangle_+ \setminus \triangle_+^1) \times 2 + \sharp\triangle_+^1 + 1 & (\mathrm{rank}(G/K) = 2) \\ \sharp(\triangle_+ \setminus \triangle_+^1) \times 2 + \sharp\triangle_+^1 & (\mathrm{rank}(G/K) = 1). \end{cases}$$

If N is a complex hyperbolic space of complex dimension bigger than one, we have $\sharp\mathrm{Spec}\,A \leq 3$ because of $\sharp(\triangle_+ \setminus \triangle_+^1) = \sharp\triangle_+^1 = 1$. Also, if N is a quarternionic hyperbolic space or the Cayley hyperbolic plane, then we have $g \leq 4$ because of $\sharp(\triangle_+ \setminus \triangle_+^1) = 2$ and $\sharp\triangle_+^1 = 0$. These facts have already been shown by J. Berndt ([B1,2]) without the assumption that M has no focal point of non-Euclidean type on $N(\infty)$.

(ii) For each irreducible symmetric space G/K of non-compact type, the number

$$m_{G/K} := \begin{cases} \sharp(\triangle_+ \setminus \triangle_+^1) \times 2 + \sharp\triangle_+^1 + 2 & \mathrm{rank}(G/K) \geq 3) \\ \sharp(\triangle_+ \setminus \triangle_+^1) \times 2 + \sharp\triangle_+^1 + 1 & \mathrm{rank}(G/K) = 2) \\ \sharp(\triangle_+ \setminus \triangle_+^1) \times 2 + \sharp\triangle_+^1 & \mathrm{rank}(G/K) = 1) \end{cases}$$

is given as in Tables 1 and 2.

Type	G/K	$\sharp\triangle_+$	$\sharp\triangle_+^1$	$m_{G/K}$	$\dim M$
(AI)	$SL(n,\mathbf{R})/SO(n)\ (n\geq 3)$	$\frac{n(n-1)}{2}$	$\frac{n(n-1)}{2}$	$\begin{cases} \frac{n(n-1)}{2}+2 & (n\geq 4) \\ 4 & (n=3) \end{cases}$	$\frac{n^2+n-4}{2}$
(AII)	$SU^*(2n)/Sp(n)\ (n\geq 3)$	$\frac{n(n-1)}{2}$	0	$\begin{cases} n^2-n+2 & (n\geq 4) \\ 7 & (n=3) \end{cases}$	$2n^2-n-2$
(AIII)	$SU(p,q)/S(U(p)\times U(q))$ $(1\leq p<q)$	p^2+p	p	$\begin{cases} 2p^2+p+2 & (p\geq 3) \\ 11 & (p=2) \\ 3 & (p=1) \end{cases}$	$2pq-1$
	$SU(p,p)/S(U(p)\times U(p))$ $(p\geq 2)$	p^2	p	$\begin{cases} 2p^2-p+2 & (p\geq 3) \\ 7 & (p=2) \end{cases}$	$2p^2-1$
(BDI)	$SO_0(p,q)/SO(p)\times SO(q)$ $(3\leq p<q)$	p^2	$\begin{cases} p^2 \\ p(p-1) \end{cases}$	$p^2+2 \quad (q-p=0,1)$ $p^2+p+2 \quad (q-p\geq 2)$	$pq-1$
	$SO_0(2,q)/SO(2)\times SO(q)$ $(2<q)$	4	$\begin{cases} 4 \\ 2 \end{cases}$	$5 \quad (q=2,3)$ $7 \quad (q\geq 4)$	$2q-1$
	$SO_0(1,q)/SO(1)\times SO(q)$	1	$\begin{cases} 1 \\ 0 \end{cases}$	$1 \quad (q=2)$ $2 \quad (q\geq 3)$	$q-1$
(DIII)	$SO^*(2n)/U(n)$ $(n\geq 6)$	$\begin{cases} \frac{n^2-1}{4} \\ \frac{n^2}{4} \end{cases}$	$\frac{n-1}{2}$ $\frac{n}{2}$	$\frac{n^2-n+4}{2} \quad (n:\mathrm{odd})$ $\frac{n^2-n+4}{2} \quad (n:\mathrm{even})$	n^2-n-1
	$SO^*(2n)/U(n)$ $(n=4,5)$	$\begin{cases} \frac{n^2-1}{4} \\ \frac{n^2}{4} \end{cases}$	$\frac{n-1}{2}$ $\frac{n}{2}$	$\frac{n^2-n+2}{2} \quad (n:\mathrm{odd})$ $\frac{n^2-n+2}{2} \quad (n:\mathrm{even})$	n^2-n-1
(CI)	$Sp(n,\mathbf{R})/U(n)\ (n\geq 2)$	n^2	n^2	$\begin{cases} n^2+2 & (n\geq 3) \\ n^2+1 & (n=2) \end{cases}$	n^2+n-1
(CII)	$Sp(p,q)/Sp(p)\times Sp(q)$ $(p<q)$	$p(p+1)$	0	$\begin{cases} 2p^2+2p+2 & (p\geq 3) \\ 13 & (p=2) \\ 4 & (p=1) \end{cases}$	$4pq-1$
	$Sp(p,p)/Sp(p)\times Sp(p)$ $(p\geq 2)$	p^2	0	$\begin{cases} 2p^2+2 & (p\geq 3) \\ 9 & (p=2) \end{cases}$	$4p^2-1$
(EI)	$E_6^6/Sp(4)$	36	36	38	41
(EII)	$E_6^2/SU(6)\cdot SU(2)$	24	12	38	39
(EIII)	$E_6^{-14}/Spin(10)\cdot U(1)$	6	2	11	31
(EIV)	E_6^{-26}/F_4	3	0	7	25
(EV)	$E_7^7/(SU(8)/\{\pm 1\})$	63	63	65	69
(EVI)	$E_7^{-5}/SO'(12)\cdot SU(2)$	24	12	38	63
(EVII)	$E_7^{-25}/E_6\cdot U(1)$	9	3	17	53
(EVIII)	$E_8^8/SO'(16)$	120	120	122	127
(EIX)	$E_8^{-24}/E_7\cdot Sp(1)$	24	12	38	111
(FI)	$F_4^4/Sp(3)\cdot Sp(1)$	24	24	26	27
(FII)	$F_4^{-20}/Spin(9)$	2	0	4	15
(G)	$G_2^2/SO(4)$	6	6	7	7

Table 1.

Type	G/K	$\sharp\triangle_+$	$\sharp\triangle_+^1$	$m_{G/K}$	$\dim M$
(II-A)	$SL(n,\mathbf{C})/SU(n)$ $(n\geq 3)$	$\frac{n(n-1)}{2}$	0	$\begin{cases} n^2-n+2 & (n\geq 4) \\ 7 & (n=3) \end{cases}$	n^2-2
(II-BD)	$SO(n,\mathbf{C})/SO(n)$ $(n\geq 6)$	$\begin{cases} \frac{(n-1)^2}{4} \\ \frac{n(n-2)}{4} \end{cases}$	0 0	$\frac{n^2-2n+5}{2}$ $(n:\text{odd})$ $\frac{n^2-2n+4}{2}$ $(n:\text{even})$	$\frac{(n+1)(n-2)}{2}$
	SO(5,$\mathbf{C}$)/SO(5)	4	0	9	9
(II-C)	$Sp(n,\mathbf{C})/Sp(n)$ $(n\geq 3)$	n^2	0	$2n^2+2$	$2n^2+n-1$
(II-E_6)	$E_6^{\mathbf{c}}/E_6$	36	0	74	77
(II-E_7)	$E_7^{\mathbf{c}}/E_7$	63	0	128	132
(II-E_8)	$E_8^{\mathbf{c}}/E_8$	120	0	242	247
(II-F_4)	$F_4^{\mathbf{c}}/F_4$	24	0	50	51
(II-G_2)	$G_2^{\mathbf{c}}/G_2$	6	0	13	13

Table 2.

By using Theorem C and its proof (see Section 5), we prove the following fact.

Theorem D. *Let M be a curvature-adapted isoparametric C^ω-hypersurface in a symmetric space N of non-compact type. Assume that M has no focal point of non-Euclidean type on $N(\infty)$. Then M admits the only focal submanifold and the focal submanifold is totally geodesic. Also, M is the tube of radius r over the focal submanifold, where r is the only focal radius of M.*

As stated in (vii) of Remark 1.2, principal orbits of a Hermann type action of cohomogeneity one are curvature-adapted C^ω-isoparametric hypersurfaces having no focal point of non-Euclidean type on the ideal boundary. Hermann type actions of cohomogeneity one other than $SO(1,n)\curvearrowright SO(1,n+1)/SO(n)$ are the only singular orbit and the singular orbit is totally geodesic. Hence the principal orbits of the actions are tubes over the totally geodesic singular orbit. Hence the following problem arises naturally.

Problem. *Does any curvature-adapted isoparametric C^ω-hypersurface in a symmetric space N of non-compact type having no focal point of non-Euclidean type on $N(\infty)$ occur as a principal orbit of a Hermann type action of cohomogeneity one?*

To solve this problem, we have only to investigate whether the only totally geodesic focal submanifold of the hypersurface occurs as the singular orbit of a Hermann type action of cohomogeneity one.

In Section 2, we recall basic notions. In Section 3, we prove Theorem A. In Section 4, we define the mean curvature of a proper anti-Kaehlerian Fredholm submanifold and prepare a lemma to prove Theorem B. In Section 5, we prove Theorems B, C and D.

2 Basic notions

In this section, we recall basic notions which are used in the proof of Theorems A and B. Some of the notions is not well-known for experts of this topic. Hence we shall explain them in detail. Let M be a complete connected (oriented embbedded) hypersurface in a (general) Riemannian manifold N. If sufficiently close parallel hypersurfaces of M are of constant mean curvature, then M is called an *isoparametric hypersurface*. This notion is the hypersurface version of an isoparametric submanifold with flat section is defined in [HLO], where we note that they defined this notion without the assumption of the completeness.

Next we recall the notion of an equifocal hypersurface in a symmetric space. Let M be a complete (oriented embedded) hypersurface in a symmetric space $N = G/K$ and fix a global unit normal vector field v of M. Let γ_{v_x} be the normal geodesic of M with $\gamma'_{v_x}(0) = v_x$, where $x \in M$ and $\gamma'_{v_x}(0)$ is the velocity vector of γ_{v_x} at 0. If $\gamma_{v_x}(s_0)$ is a focal point of M along γ_{v_x}, then s_0 is called a *focal radius of M at x*. Denote by $\mathcal{FR}_{M,x}$ the set of all focal radii of M at x. If M is compact and if $\mathcal{FR}_{M,x}$ is independent of the choice of x, then it is called an *equifocal hypersurface*. This notion is the hypersurface version of an equifocal submanifold defined in [TT].

Next we recall the notion of a complex equifocal hypersurface in a symmetric space of non-compact type. Let M be a complete (oriented embedded) hypersurface in a symmetric space $N = G/K$ of non-compact type and fix a global unit normal vector field v of M. Let $\mathfrak{g}$ be the Lie algebra of G and θ be the Cartan involution of G with $(\mathrm{Fix}\,\theta)_0 \subset K \subset \mathrm{Fix}\,\theta$, where $\mathrm{Fix}\,\theta$ is the fixed point group of θ and $(\mathrm{Fix}\,\theta)_0$ is the identity component of $\mathrm{Fix}\,\theta$. Denote by the same symbol θ the involution of $\mathfrak{g}$ induced from θ. Set $\mathfrak{p} := \mathrm{Ker}(\theta + \mathrm{id})$. The subspace $\mathfrak{p}$ is identified with the tangent space $T_{eK}N$ of N at eK, where e is the identity element of G. Let M be a complete (oriented embedded) hypersurface in N. Fix a global unit normal vector field v of M. Denote by A the shape operator of M (for v). Take $X \in T_xM$ ($x = gK$). The M-Jacobi field Y along γ_x with $Y(0) = X$ (hence $Y'(0) = -A_xX$) is given by

$$Y(s) = (P_{\gamma_x|_{[0,s]}} \circ (D^{co}_{sv_x} - sD^{si}_{sv_x} \circ A_x))(X),$$

where $P_{\gamma_x|_{[0,s]}}$ is the parallel translation along $\gamma_x|_{[0,s]}$, $D^{co}_{sv_x}$ (resp. $D^{si}_{sv_x}$) is given by

$$D^{co}_{sv_x} = g_* \circ \cos(\mathbf{i}\mathrm{ad}(sg_*^{-1}v_x)) \circ g_*^{-1}$$
$$\left(\text{resp. } D^{si}_{sv_x} = g_* \circ \frac{\sin(\mathbf{i}\mathrm{ad}(sg_*^{-1}v_x))}{\mathbf{i}\mathrm{ad}(sg_*^{-1}v_x)} \circ g_*^{-1}\right).$$

Here ad is the adjoint representation of the Lie algebra $\mathfrak{g}$ of G. All focal radii of M at x are catched as real numbers s_0 with $\mathrm{Ker}(D^{co}_{s_0v_x} - s_0D^{si}_{s_0v_x} \circ A_x) \neq \{0\}$. So, we [Koi2] defined the notion of a *complex focal radius of M at x* as a complex number z_0 with $\mathrm{Ker}(D^{co}_{z_0v_x} - z_0D^{si}_{z_0v_x} \circ A^{\mathbf{c}}_x) \neq \{0\}$, where $D^{co}_{z_0v_x}$ (resp. $D^{si}_{z_0v_x}$) is a $\mathbf{C}$-linear transformation of $(T_xN)^{\mathbf{c}}$ defined by

$$D^{co}_{z_0v_x} = g^{\mathbf{c}}_* \circ \cos(\mathbf{i}\mathrm{ad}^{\mathbf{c}}(z_0g_*^{-1}v_x)) \circ (g^{\mathbf{c}}_*)^{-1}$$
$$\left(\text{resp. } D^{si}_{sv_x} = g^{\mathbf{c}}_* \circ \frac{\sin(\mathbf{i}\mathrm{ad}^{\mathbf{c}}(z_0g_*^{-1}v_x))}{\mathbf{i}\mathrm{ad}^{\mathbf{c}}(z_0g_*^{-1}v_x)} \circ (g^{\mathbf{c}}_*)^{-1}\right),$$

where $g_*^{\mathbf{c}}$ (resp. $\mathrm{ad}^{\mathbf{c}}$) is the complexification of g_* (resp. ad). Also, we call $\mathrm{Ker}(D^{co}_{z_0v_x} - z_0D^{si}_{z_0v_x} \circ A^{\mathbf{c}}_x)$ the *foccal space* of the complex focal radius z_0 and its complex dimension the *multiplicity* of the complex focal radius z_0, In [Koi3], it was shown that, in the case where M is of class C^ω, complex focal radii of M at x indicate the positions of focal points of the extrinsic complexification $M^{\mathbf{c}}(\hookrightarrow G^{\mathbf{c}}/K^{\mathbf{c}})$ of M along the complexified geodesic $\gamma^{\mathbf{c}}_{v_x}$, where $G^{\mathbf{c}}/K^{\mathbf{c}}$ is the anti-Kaehlerian symmetric space associated with G/K. See [Koi3] (also [Koi9]) about the detail of the definition of the extrinsic complexification. Denote by $\mathcal{CFR}_x$ the set of all complex focal radii of M at x. If $\mathcal{CFR}_x$ is independent of the choice of x, then M is called a *complex equifocal hypersurface*. Here we note that we should call such a hypersurface an equi-complex focal hypersurface but, for simplicity, we call it a complex equifocal hypersurface. This notion is the hypersurface version of a complex equifocal submanifold defined in [Koi2].

Next we recall the notion of an isoaparmetric hypersurface in a (separable) Hilbert space introduced by C.L. Terng in [Te2], where we note that she introduced the notion of an isoparametric submanifold in the space in general. Let M be a complete (embedded) hypersurface in a (separable) Hilbert space V. If the normal exponential map $\exp^\perp$ of M is a Fredholm map and the restriction of $\exp^\perp$ to unit disk normal bundle of M is proper, then M is called a *proper Fredholm hypersurface*. Let v be a unit normal vector field of a proper Fredholm hypersurface M and A the shape operator of M for v. For each $x \in M$, A_x is compact and self-adjoint. Hence the spectrum of A_x is described as $\{0\} \cup \{\lambda^x_i \,|\, i = 1, 2, \cdots\}$ ($|\lambda^x_i| > |\lambda^x_{i+1}|$ or $\lambda^x_i = -\lambda^x_{i+1} > 0$). We call λ^x_i the i-th *principal curvature* of M at x. If the series $\sum_{i=1}^{\infty} \lambda^x_i$ exists, then we call it the *mean curvature of M at x* and denote it by $\mathrm{Tr}\, A_x$, where we promise $\lambda_i = 0$ ($i \geq k+1$) if the cardinal number of the spectrum other than zero of A_x is equal to k. If $\mathrm{Tr}\, A_x = 0$ for any $x \in M$, then M is said to be *minimal* (or *formal minimal*). Assume that the number (which may be ∞) of distinct principal curvatures of M at x is independent of x. Then we can define functions λ_i ($i = 1, 2, \cdots$) on M by assigning the i-th principal curvature of M at x to each $x \in M$. We call this function λ_i the i-th *principal curvature function of M*. If the principal curvature functions λ_i's of M are constant over M, then M is called an *isoparametric hypersurface* and the constant λ_i is called the i-th principal curvature of M. Assume that M is an isoparametric hypersurface. Let λ_i be the i-th principal curvature of M and set $E_i := \mathrm{Ker}(A - \lambda_i\, \mathrm{id})$. This distribution E_i is called the *curvature distribution for λ_i*. Note that the tangent space T_xM of M at x is equal to the closure of the orthogonal direct sum of $(E_i)_x$'s.

Next we recall the notion of the parallel transport map for a compact semi-simple Lie group. Let G be a semi-simple compact Lie group and $\mathfrak{g}$ be its Lie algebra. Let $\langle\,,\,\rangle$ be an Ad(G)-invariant inner product of $\mathfrak{g}$. Let $H^0([0,1], \mathfrak{g})$ be the Hilbert space of all L^2-integrable paths $u : [0,1] \to \mathfrak{g}$, where the L^2-inner product is defined by $\langle u, v\rangle_0 := \int_0^1 \langle u(t), v(t)\rangle dt$, and $H^1([0,1], \mathfrak{g})$ be the space of all absolutely continuous paths $u : [0,1] \to \mathfrak{g}$ such that $u' \in H^0([0,1], \mathfrak{g})$. Also, let $H^1([0,1], G)$ be the Hilbert Lie group of all absolutely continuous paths $g : [0,1] \to G$ such that g' is square integrable, that is, $\langle g_*^{-1}g', g_*^{-1}g'\rangle_0$ is finite. Note that the tangent space $T_gH^1([0,1], G)$ of $H^1([0,1], G)$ at g is given by $\{vg \,|\, v \in H^1([0,1], \mathfrak{g})\}$. Define a map $\phi : H^0([0,1], \mathfrak{g}) \to G$ by $\phi(u) = g_u(1)$ ($u \in H^0([0,1], \mathfrak{g})$), where g_u is the element of $H^1([0,1], G)$ satisfying $g_u(0) = e$ and $g_{u*}^{-1}g_u' = u$. This map ϕ is called the *parallel transport map* (*from* 0 *to* 1). It is shown that

ϕ is a Riemannian submersion. Let M be a compact (embedded) hypersurface in a simply connected symmetric space G/K of compact type and $\widetilde{M}$ any component of the inverse image $(\pi\circ\phi)^{-1}(M)$, where π is the natural projection of G onto G/K. Terng-Thorbergsson ([TT]) showed the following fact.

Fact 1. *M is equifocal if and only if $\widetilde{M}$ is isoparametric.*

According to the proof of Theorem 4.1 in [Koi1], the following fact holds.

Fact 2. *Assume that M is curvature-adapted. Then we have $\mathrm{Tr}\,\widetilde{A}_u = \mathrm{Tr}\,A_{(\pi\circ\phi)(u)}$ $(u \in \widetilde{M})$, where A (resp. $\widetilde{A}$) is the shape operator of M (resp. $\widetilde{M}$).*

Next we recall the notion of an anti-Kaehlerian equifocal hypersurface in an anti-Kaehlerian symmetric space. Let J be a parallel complex structure on an even dimensional pseudo-Riemannian manifold $(M, \langle\ ,\ \rangle)$ of half index. If $\langle JX, JY\rangle = -\langle X, Y\rangle$ holds for every X, $Y \in TM$, then $(M, \langle\ ,\ \rangle, J)$ is called an *anti-Kaehlerian manifold.* Let $N = G/K$ be a symmetric space of non-compact type and $G^{\mathbf{c}}/K^{\mathbf{c}}$ the anti-Kaehlerian symmetric space associated with G/K. See [Koi3] about the anti-Kaehlerian structure of $G^{\mathbf{c}}/K^{\mathbf{c}}$. Let f be an isometric immersion of an anti-Kaehlerian manifold $(M, \langle\ ,\ \rangle, J)$ into $G^{\mathbf{c}}/K^{\mathbf{c}}$. If $\widetilde{J}\circ f_* = f_*\circ J$, then M is called an *anti-Kaehlerian submanifold* immersed by f. Let A be the shape tensor of M. We have $A_{\widetilde{J}v}X = A_v(JX) = J(A_vX)$, where $X \in TM$ and $v \in T^{\perp}M$. If $A_vX = aX + bJX$ $(a, b \in \mathbf{R})$, then X is called a *J-eigenvector for $a + b\mathbf{i}$*. Let $\{e_i\}_{i=1}^n$ be an orthonormal system of T_xM such that $\{e_i\}_{i=1}^n \cup \{Je_i\}_{i=1}^n$ is an orthonormal base of T_xM. We call such an orthonormal system $\{e_i\}_{i=1}^n$ a *J-orthonormal base* of T_xM. If there exists a J-orthonormal base consisting of J-eigenvectors of A_v, then we say that *A_v is diagonalizable with respect to an J-orthonormal base.* Then we set $\mathrm{Tr}_J A_v := \sum\limits_{i=1}^{n}\lambda_i$ as $A_ve_i = (\mathrm{Re}\,\lambda_i)e_i + (\mathrm{Im}\,\lambda_i)Je_i$ $(i = 1, \cdots, n)$. We call this quantity the J-trace of A_v. If, for each unit normal vector $v \in M$, the shape operator A_v is diagonalizable with respect to a J-orthonormal tangent base, if the normal Jacobi operator $R(v)$ preserves the tangent space T_xM (x : the base point of v) invariantly and if A_v and $R(v)$ commute, then we call M a *curvature-adapted anti-Kaehlerian submanifold*, where R is the curvature tensor of $G^{\mathbf{c}}/K^{\mathbf{c}}$. Assume that M is an anti-Kaehlerian hypersurface (i.e., $\mathrm{codim}\,M = 2$) and that it is orientable. Denote by $\exp^{\perp}$ the normal exponential map of M. Fix a global parallel orthonormal normal base $\{v, Jv\}$ of M. If $\exp^{\perp}(av_x + bJv_x)$ is a focal point of (M, x), then we call the complex number $a + b\mathbf{i}$ a *complex focal radius along the geodesic γ_{v_x}*. Assume that the number (which may be 0 and ∞) of distinct complex focal radii along the geodesic γ_{v_x} is independent of the choice of $x \in M$. Furthermore assume that the number is not equal to 0. Let $\{r_{i,x} \mid i = 1, 2, \cdots\}$ be the set of all complex focal radii along γ_{v_x}, where $|r_{i,x}| < |r_{i+1,x}|$ or "$|r_{i,x}| = |r_{i+1,x}|$ & $\mathrm{Re}\,r_{i,x} > \mathrm{Re}\,r_{i+1,x}$" or "$|r_{i,x}| = |r_{i+1,x}|$ & $\mathrm{Re}\,r_{i,x} = \mathrm{Re}\,r_{i+1,x}$ & $\mathrm{Im}\,r_{i,x} = -\mathrm{Im}\,r_{i+1,x} < 0$". Let r_i $(i = 1, 2, \cdots)$ be complex-valued functions on M defined by assigning $r_{i,x}$ to each $x \in M$. We call this function r_i the *i-th complex focal radius function for $\widetilde{v}$*. If the number of distinct complex focal radii along γ_{v_x} is independent of the choice of $x \in M$, complex focal radius functions for v are constant on M and they have constant multiplicity, then M is called an *anti-Kaehlerian equifocal hypersurface*. We ([Koi3]) showed the following fact.

Fact 3. *Let M be a complete (embedded) C^{ω}-hypersurface in G/K. Then M is complex*

equifocal if and only if $M^{\mathbf{c}}$ is anti-Kaehler equifocal.

Next we recall the notion of an anti-Kaehlerian isoparametric hypersurface in an infinite dimensional anti-Kaehlerian space. Let f be an isometric immersion of an anti-Kaehlerian Hilbert manifold $(M, \langle\ ,\ \rangle, J)$ into an infinite dimensional anti-Kaehlerian space $(V, \langle\ ,\ \rangle, \widetilde{J})$. See Section 5 of [Koi3] about the definitions of an anti-Kaehlerian Hilbert manifold and an infinite dimensional anti-Kaehlerian space. If $\widetilde{J} \circ f_* = f_* \circ J$ holds, then we call M an *anti-Kaehlerian Hilbert submanifold in* $(V, \langle\ ,\ \rangle, \widetilde{J})$ *immersed by* f. If M is of finite codimension and there exists an orthogonal time-space decomposition $V = V_- \oplus V_+$ such that $\widetilde{J}V_\pm = V_\mp$, $(V, \langle\ ,\ \rangle_{V_\pm})$ is a Hilbert space, the distance topology associated with $\langle\ ,\ \rangle_{V_\pm}$ coincides with the original topology of V and, for each $v \in T^\perp M$, the shape operator A_v is a compact operator with respect to $f^*\langle\ ,\ \rangle_{V_\pm}$, then we call M a *anti-Kaehlerian Fredholm submanifold* (rather than *anti-Kaehlerian Fredholm Hilbert submanifold*). Let $(M, \langle\ ,\ \rangle, J)$ be an orientable anti-Kaehlerian Fredholm hypersurface in an anti-Kaehlerian space $(V, \langle\ ,\ \rangle, \widetilde{J})$ and A be the shape tensor of $(M, \langle\ ,\ \rangle, J)$. Fix a global unit normal vector field v of M. If there exists $X(\neq 0) \in T_xM$ with $A_{v_x}X = aX + bJX$, then we call the complex number $a + b\mathbf{i}$ a *J-eigenvalue of A_{v_x}* (or a *complex principal curvature of M at x*) and call X a *J-eigenvector of A_{v_x} for $a + b\mathbf{i}$*. Here we note that this relation is rewritten as $A^{\mathbf{c}}_{v_x}X^{(1,0)} = (a + b\mathbf{i})X^{(1,0)}$, where $X^{(1,0)}$ is as above. Also, we call the space of all J-eigenvectors of A_{v_x} for $a + b\sqrt{-1}$ a *J-eigenspace of A_{v_x} for $a + b\mathbf{i}$*. We call the set of all J-eigenvalues of A_{v_x} the *J-spectrum of A_{v_x}* and denote it by $\mathrm{Spec}_J A_{v_x}$. $\mathrm{Spec}_J A_{v_x} \setminus \{0\}$ is described as follows:

$$\mathrm{Spec}_J A_{v_x} \setminus \{0\} = \{\lambda_i \,|\, i = 1, 2, \cdots\}$$

$$\left(\begin{array}{c} |\lambda_i| > |\lambda_{i+1}| \text{ or } "|\lambda_i| = |\lambda_{i+1}| \;\&\; \mathrm{Re}\,\lambda_i > \mathrm{Re}\,\lambda_{i+1}" \\ \text{or } "|\lambda_i| = |\lambda_{i+1}| \;\&\; \mathrm{Re}\,\lambda_i = \mathrm{Re}\,\lambda_{i+1} \;\&\; \mathrm{Im}\,\lambda_i = -\mathrm{Im}\,\lambda_{i+1} > 0" \end{array}\right).$$

Also, the J-eigenspace for each J-eigenvalue of A_{v_x} other than 0 is of finite dimension. We call the J-eigenvalue λ_i the *i-th complex principal curvature of M at x*. Assume that the number (which may be ∞) of distinct complex principal curvatures of M is constant over M. Then we can define functions $\widetilde{\lambda}_i$ $(i = 1, 2, \cdots)$ on M by assigning the i-th complex principal curvature of M at x to each $x \in M$. We call this function $\widetilde{\lambda}_i$ the *i-th complex principal curvature function* of M. If the number of distinct complex principal curvatures of M is constant over M, each complex principal curvature function is constant over M and it has constant multiplicity, then we call M an *anti-Kaehler isoparametric hypersurface*. Let $\{e_i\}_{i=1}^\infty$ be an orthonormal system of $(T_xM, \langle\ ,\ \rangle_x)$. If $\{e_i\}_{i=1}^\infty \cup \{Je_i\}_{i=1}^\infty$ is an orthonormal base of T_xM, then we call $\{e_i\}_{i=1}^\infty$ a *J-orthonormal base*. If there exists a J-orthonormal base consisting of J-eigenvectors of A_{v_x}, then A_{v_x} is said to *be diagonalized with respect to the J-orthonormal base*. If M is anti-Kaehlerian isoparametric and, for each $x \in M$, the shape operator A_{v_x} is diagonalized with respect to an J-orthonormal base, then we call M a *proper anti-Kaehlerian isoparametric hypersurface*.

In [Koi3], we defined the notion of the parallel transport map for the complexification $G^{\mathbf{c}}$ of a semi-simple Lie group G as an anti-Kaehlerian submersion of an infinite dimensional anti-Kaehlerian space $H^0([0,1], \mathfrak{g}^{\mathbf{c}})$ onto $G^{\mathbf{c}}$. See [Koi3] in detail. Let G/K be a symmetric space of non-compact type and $\phi : H^0([0,1], \mathfrak{g}^{\mathbf{c}}) \to G^{\mathbf{c}}$ the parallel transport map for $G^{\mathbf{c}}$ and $\pi : G^{\mathbf{c}} \to G^{\mathbf{c}}/K^{\mathbf{c}}$ the natural projection. We [Koi3] showed the following fact.

Fact. 4. *Let M be a complete anti-Kaehlerian hypersurface in an anti-Kaehlerian symmetric space $G^{\mathbf{c}}/K^{\mathbf{c}}$. Then M is anti-Kaehlerian equifocal if and only if each component of $(\pi\circ\phi)^{-1}(M)$ is anti-Kaehlerian isoparametric.*

Next we recall the notion of a focal point of non-Euclidean type on the ideal boundary $N(\infty)$ of a hypersurface M in a Hadamard manifold N which was introduced in [Koi11] for a submanifold of general codimension. Assume that M is orientable. Denote by $\widetilde{\nabla}$ the Levi-Civita connection of N and A the shape operator of M for a unit normal vector field v. Let $\gamma_{v_x}:[0,\infty)\to N$ be the normal geodesic of M of direction v_x. If there exists a M-Jacobi field Y along γ_v satisfying $\lim\limits_{t\to\infty}\frac{\|Y_t\|}{t}=0$, then we call $\gamma_v(\infty)\,(\in N(\infty))$ a *focal point* of M *on the ideal boundary* $N(\infty)$ *along* γ_{v_x}, where $\gamma_{v_x}(\infty)$ is the asymptotic class of γ_{v_x}. Also, if there exists a M-Jacobi field Y along γ_{v_x} satisfying $\lim\limits_{t\to\infty}\frac{\|Y(t)\|}{t}=0$ and $\mathrm{Sec}(v_x,Y(0))\neq 0$, then we call $\gamma_{v_x}(\infty)$ a *focal point of non-Euclidean type of M on $N(\infty)$ along* γ_{v_x}, where $\mathrm{Sec}(v_x,Y(0))$ is the sectional curvature for the 2-plane spanned by v_x and $Y(0)$. If, for any point x of M, $\gamma_{v_x}(\infty)$and $\gamma_{-v_x}(\infty)$ is not a focal point of non-Euclidean type of M on $N(\infty)$, then we say that *M has no focal point of non-Euclidean type on the ideal boundary* $N(\infty)$. According to Theorem 1 of [Koi3] and Theorem A of [Koi11], we have the following fact.

Fact 5. *Let M be a curvature-adapted and isoparametric C^ω-hypersurface in a symmetric space $N:=G/K$ of non-compact type. Then the following conditions* (i) *and* (ii) *are equivalent:*

(i) *M has no focal point of non-Euclidean type on the ideal boundary $N(\infty)$.*

(ii) *each component of $(\pi\circ\phi)^{-1}(M^{\mathbf{c}})$ is proper anti-Kaehlerian isoparametric,*

where π is the natural projection of $G^{\mathbf{c}}$ onto $G^{\mathbf{c}}/K^{\mathbf{c}}$ and ϕ is the parallel transport map for $G^{\mathbf{c}}$.

3 Proof of Theorem A

In this section, we first prove Theorem A. Let M be a curvature-adapted isoparametric hypersurface in a simply connected symmetric space G/K of compact type, v a unit normal vector field of M and $C(\subset T^\perp_xM)$ the Coxeter domain (i.e., the fundamental domain (containing 0) of the Coxeter group of M at x). The boundary ∂C of C consists of two points and it is described as $\partial C=\{r_1v_x,r_2v_x\}$ $(r_2<0<r_1)$. We may assume that $|r_1|\leq|r_2|$ by replacing v to $-v$ if necessary. Note that the set $\mathcal{FR}_M$ of all focal radii of M is equal to $\{kr_1+(1-k)r_2\,|\,k\in\mathbb{Z}\}$. Set $F_i:=\{\gamma_{v_x}(r_i)\,|\,x\in M\}$ $(i=1,2)$, which are all of focal submanifolds of M. The hypersurface M is the r_i-tube over F_i $(i=1,2)$. Let π be the natural projection of G onto G/K and ϕ the parallel transport map for G. Let $\widetilde{M}$ be a component of $(\pi\circ\phi)^{-1}(M)$, which is an isoparametric hypersurface in $H^0([0,1],\mathfrak{g})$. The set $\mathcal{PC}_{\widetilde{M}}$ of all principal curvatures other than zero of $\widetilde{M}$ is equal to $\{\frac{1}{kr_1+(1-k)r_2}\,|\,k\in\mathbb{Z}\}$. Set $\lambda_{2k-1}:=\frac{1}{kr_1+(1-k)r_2}$ $(k=1,2,\cdots)$ and $\lambda_{2k}:=\frac{1}{-(k-1)r_1+kr_2}$ $(k=1,2,\cdots)$. Then we have $|\lambda_{i+1}|<|\lambda_i|$ or $\lambda_i=-\lambda_{i+1}>0$ for any $i\in\mathbb{N}$. Denote by m_i the multiplicity of λ_i. Denote by A (resp. $\widetilde{A}$) the shape operator of M for v (resp. $\widetilde{M}$ for v^L), where v^L is the horizontal lift of v to $\widetilde{M}$ with respect to $\pi\circ\phi$. Fix $r_0\in\mathcal{FR}_M$. The focal map $f_{r_0}:M\to G/K$ is defined by $f_{r_0}(x):=\gamma_{v_x}(r_0)$ $(x\in M)$. Let $F:=f_{r_0}(M)$, which is

either F_1 or F_2. Denote by A^F the shape tensor of F. Let ψ_t be the geodesic flow of G/K. Then we have the following fact.

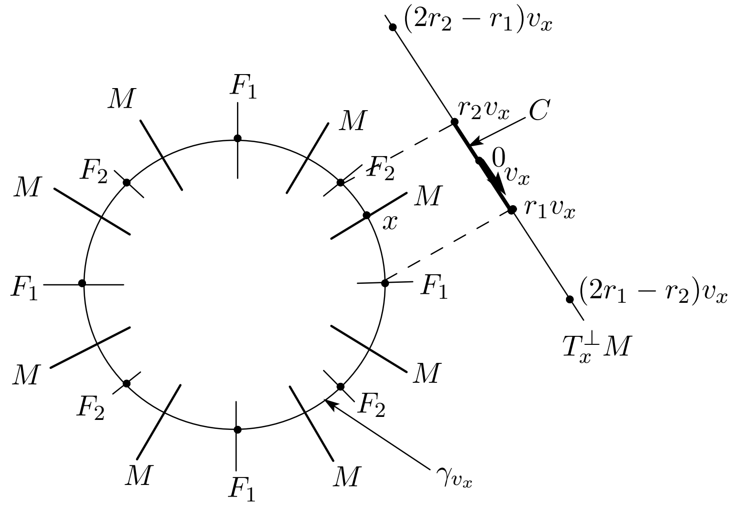


Fig. 2.

Lemma 3.1. *For any $x \in M$, the following relation holds:*

$$\mathrm{Tr}\, A^F_{\psi_{r_0}(v_x)} = -\sum_{(\lambda,\mu)\in S^x_{r_0}} \frac{\mu + \lambda\tau_{r_0}(\mu)}{\lambda - \tau_{r_0}(\mu)} \times m_{\lambda,\mu},$$

where $S^x_{r_0}$ and $m_{\lambda,\mu}$ are as in the statement of Theorem A.

Proof. Let $S_x := \{(\lambda,\mu) \in \mathrm{Spec}A_x \times \mathrm{Spec}R(v_x) \,|\, \mathrm{Ker}(A_x - \lambda I) \cap \mathrm{Ker}(R(v_x) - \mu I) \neq \{0\}\}$. Since M is curvature adapted, we have $T_xM = \bigoplus_{(\lambda,\mu)\in S_x} (\mathrm{Ker}(A_x - \lambda I) \cap \mathrm{Ker}(R(v_x) - \mu I))$. Defone a distribution D on M by $D_x := \bigoplus_{(\lambda,\mu)\in S^x_{r_0}} (\mathrm{Ker}(A_x - \lambda I) \cap \mathrm{Ker}(R(v_x) - \mu I))$ and $D^\perp$ the orthogonal complementary distribution of D in TM. Let $X \in \mathrm{Ker}(A_x - \lambda I) \cap \mathrm{Ker}(R(v_x) - \mu I)$ ($(\lambda,\mu) \in S^x_{r_0}$) and Y be the Jacobi field along $\gamma_{r_0v_x}$ with $Y(0) = X$ and $Y'(0) = -A_{r_0v_x}X$ $(= -r_0\lambda X)$. This Jacobi field Y is described as

$$Y(s) = \left(\cos(sr_0\sqrt{\mu}) - \frac{\lambda\sin(sr_0\sqrt{\mu})}{\sqrt{\mu}}\right) P_{\gamma_{r_0v}|_{[0,s]}}(X).$$

Since $Y(1) = f_{r_0*}X$, we have

$$f_{r_0*}X = \left(\cos(r_0\sqrt{\mu}) - \frac{\lambda\sin(r_0\sqrt{\mu})}{\sqrt{\mu}}\right) P_{\gamma_{r_0v_x}}(X), \tag{3.1}$$

which is not equal to 0 because of $(\lambda,\mu) \in S^x_{r_0}$. From this relation, we have $T_{f_{r_0}(x)}F =$

$P_{\gamma_{r_0 v_x}}(D)$. On the other hand, we have

$$
\begin{aligned}
\widetilde{\nabla}_{f_{r_0 *}X}\psi_{r_0}(v_x) &= \frac{1}{r_0}Y'(1) \\
&= -\left(\sqrt{\mu}\sin(r_0\sqrt{\mu}) + \lambda\cos(r_0\sqrt{\mu})\right) P_{\gamma_{r_0 v_x}}(X).
\end{aligned} \tag{3.2}
$$

From (3.1) and (3.2), we have

$$
A^F_{\psi_{r_0}(v_x)} f_{r_0 *}X = -\frac{\mu + \lambda\tau_{r_0}(\mu)}{\lambda - \tau_{r_0}(\mu)} f_{r_0 *}X. \tag{3.3}
$$

The desired relation follows from this relation. q.e.d.

Set $\kappa(\lambda,\mu) := -\frac{\mu+\lambda\tau_{r_0}(\mu)}{\lambda-\tau_{r_0}(\mu)}$ $((\lambda,\mu)\in S_{r_0})$. Next we prepare the following lemma.

Lemma 3.2. *Let* $(\lambda_1,\mu_1)\in S^x_{r_0}$. *Then we have*

(i) $(\exp_G r_0 v_x)^{-1}_*(\psi_{r_0}(v_x)) = v_x$, *where* $\exp_G$ *is the exponential map of* G,

(ii) $(\exp_G r_0 v_x)^{-1}_*\left(\mathrm{Ker}(A^F_{\psi_{r_0}(v_x)} - \kappa(\lambda_1,\mu_1)I)\right)$
$= \bigoplus_{(\lambda,\mu)\in S^x_{r_0}(\lambda_1,\mu_1)} (\mathrm{Ker}(A_{v_x} - \lambda I)\cap \mathrm{Ker}(R(v_x)-\mu I))$,

where $S^x_{r_0}(\lambda_1,\mu_1) = \{(\lambda,\mu)\in S^x_{r_0}\,|\,\kappa(\lambda,\mu)=\kappa(\lambda_1,\mu_1)\}$.

Proof. The relation in (i) is trivial. We shall show the statement (ii). Let $(\lambda,\mu)\in S^x_{r_0}(\lambda_1,\mu_1)$. The restriction $f_{r_0 *}|_{\mathrm{Ker}(A_x-\lambda I)\cap\mathrm{Ker}(R(v_x)-\mu I)}$ of $f_{r_0 *}$ is equal to $P_{\gamma_{r_0 v_x}}|_{\mathrm{Ker}(A_x-\lambda I)\cap\mathrm{Ker}(R(v_x)-\mu I)}$ up to constant multiple by (3.1). Also, we have $P_{\gamma_{r_0 v_x}} = (\exp_G r_0 v_x)_*$. These facts together with (3.3) deduce the relation in (ii). q.e.d.

By using these lemmas, we prove Theorem A. According to Lemma 3.1, we have only to show $\mathrm{Tr}\,A^F_{\psi_{r_0}(v_x)} = 0$. In the case where M is homogeneous, then this relation is shown by imitating the process of the proof of Corollary 1.1 of [HL]. Also, in the case where G/K is of rank one, we can show this relation in a comparatively simple method as follows.

Simple proof of Theorem A in rank one case. We have only to show $\mathrm{Tr}\,A^F_{\psi_{r_0}(v_x)} = 0$. Assume that G/K is of rank one. Define a linear function $\Phi : T^\perp_{f_{r_0}(x)}F \to \mathbb{R}$ by $\Phi(w) = \mathrm{Tr}\,A^F_w$ $(w\in T^\perp_{f_{r_0}(x)}F)$. The focal map f_{r_0} is a submersion of M onto F and the fibres of f_{r_0} are integral manifolds of $D^\perp$, where $D^\perp$ is as in the proof of Lemma 3.1. Let L be the integral manifold of $D^\perp$ through $x_0(\in M)$ and set $Q := \{\psi_{r_0}(v_x)\,|\,x\in L\}$, which is a hypersurface without geodesic point in $T^\perp_{f_{r_0}(x_0)}F$, that is, it is not contained in any affine hyperplane of $T^\perp_{f_{r_0}(x_0)}F$. According to Lemma 3.1, we have

$$
\Phi(\psi_{r_0}(v_x)) = -\sum_{(\lambda,\mu)\in S^x_{r_0}} \frac{\mu+\lambda\tau_{r_0}(\mu)}{\lambda-\tau_{r_0}(\mu)} \times m_{\lambda,\mu}.
$$

Let $(\widetilde{\lambda},\widetilde{\mu})$ be a pair of continuous functions on L such that $(\widetilde{\lambda}(x),\widetilde{\mu}(x))\in S^x_{r_0}$ for any $x\in L$. Since G/K is of rank one, μ is constant on L. The focal radius having $\mathrm{Ker}(A_x -$

$\widetilde{\lambda}(x)\,I) \cap \mathrm{Ker}(R(v_x) - \mu(x)\,I)$ as a part of the focal space is the real number s_0 satisfying $\mathrm{Ker}(D^{co}_{s_0 v_x} - s_0 D^{si}_{s_0 v_x} \circ A_x)|_{\mathrm{Ker}(A_{v_x} - \widetilde{\lambda}(x)\,I) \cap \mathrm{Ker}(R(v_x) - \widetilde{\mu}(x)\,I)} \neq \{0\}$, that is, it is equal to $\frac{1}{\sqrt{\widetilde{\mu}(x)}} \arctan \frac{\sqrt{\widetilde{\mu}(x)}}{\widetilde{\lambda}(x)}$, which is independent of the choice of $x \in L$ by the isoparametricness (hence equifocality) of M. Hence $\widetilde{\lambda}$ is constant on L. Therefore Φ is constant along Q. Furthermore, this fact together with the linearity of Φ imply $\Phi \equiv 0$. In particular, we have $\mathrm{Tr}\, A^F_{\psi_{r_0}(v_{x_0})} = 0$. q.e.d.

In general case, we prove Theorem A.

Proof of Theorem A (general case). According to Lemma 3.1, we have only to show $\mathrm{Tr}\, A^F_{\psi_{r_0}(v_{x_0})} = 0$ $(x_0 \in M)$. We shall show this relation by showing the minimality of the focal submanifold of $(\pi \circ \phi)^{-1}(M)$, where ϕ $(: H^0([0,1], \mathfrak{g}) \to G)$ is the parallel transport map for G and π is the natural projection of G onto G/K. Let $\widetilde{M}$ be a component of $(\pi \circ \phi)^{-1}(M)$. Let v^L the horizontal lift of v to $\widetilde{M}$. Since $\pi \circ \phi$ is a Riemannian submersion, the focal radii of M are those of $\widetilde{M}$. Let r_0 be a focal radius of M (hence $\widetilde{M}$). The focal map $\widetilde{f}_{r_0}$ for r_0 is defined by $\widetilde{f}_{r_0}(x) = x + r_0 v^L_x$ $(x \in \widetilde{M})$. Set $\widetilde{F} := \widetilde{f}_{r_0}(\widetilde{M})$. Denote by $\widetilde{A}$ the shape operator of $\widetilde{M}$ for v^L and $A^{\widetilde{F}}$ the shape tensor of $\widetilde{F}$. Let $\mathrm{Spec}\, \widetilde{A}_{\hat{0}} \setminus \{0\} = \{\lambda_i \,|\, i = 1, 2, \cdots\}$ ("$|\lambda_i| > |\lambda_{i+1}|$" or "$\lambda_i = -\lambda_{i+1} > 0$"). The set of all focal radii of $\widetilde{M}$ is equal to $\{\frac{1}{\lambda_i} \,|\, i = 1, 2, \cdots\}$. We have $r_0 = \frac{1}{\lambda_{i_0}}$ for some i_0. Define a distribution $\widetilde{D}_i$ $(i = 0, 1, 2, \cdots)$ on $\widetilde{M}$ by $(\widetilde{D}_0)_u := \mathrm{Ker} \widetilde{A}_u$ and $(\widetilde{D}_i)_u := \mathrm{Ker}(\widetilde{A}_u - \lambda_i I)$ $(i = 1, 2, \cdots)$, where $u \in \widetilde{M}$. Since M is isoparametric (hence equifocal), $\widetilde{M}$ is isoparametric. Therefore, we have $T\widetilde{M} = \overline{\widetilde{D}_0 \oplus (\underset{i}{\oplus} \widetilde{D}_i)}$ and $\mathrm{Spec}\, \widetilde{A}_u$ is independent of the choice of $u \in \widetilde{M}$. Take $u_0 \in \widetilde{M}$ with $(\pi \circ \phi)(u_0) = x_0$. Let $X_i \in (\widetilde{D}_i)_{u_0}$ $(i \neq i_0)$ and $X_0 \in (\widetilde{D}_0)_{u_0}$. Then we have $\widetilde{f}_{r_0 *} X_i = (1 - r_0 \lambda_i) X_i$ and $\widetilde{f}_{r_0 *} X_0 = X_0$. Hence we have $T_{\widetilde{f}_{r_0}(u_0)} \widetilde{F} = (\widetilde{D}_0)_{u_0} \oplus (\underset{i \neq i_0}{\oplus} (\widetilde{D}_i)_{u_0})$ and $\mathrm{Ker}(\widetilde{f}_{r_0})_{*u_0} = (\widetilde{D}_{i_0})_{u_0}$, which implies that $\widetilde{D}_{i_0}$ is integrable. On the other hand, we have $A^{\widetilde{F}}_{\widetilde{\psi}_{r_0}(v^L_{u_0})} \widetilde{f}_{r_0 *} X_i = \lambda_i X_i$ and $A^{\widetilde{F}}_{\widetilde{\psi}_{r_0}(v^L_{u_0})} \widetilde{f}_{r_0 *} X_0 = 0$, where $\widetilde{\psi}$ is the geodesic flow of $H^0([0,1], \mathfrak{g})$. Therefore, we obtain $A^{\widetilde{F}}_{\widetilde{\psi}_{r_0}(v^L_{u_0})} \widetilde{f}_{r_0 *} X_i = \dfrac{\lambda_i \lambda_{i_0}}{\lambda_{i_0} - \lambda_i} \widetilde{f}_{r_0 *} X_i$. Hence we have $\mathrm{Tr}\, A^{\widetilde{F}}_{\widetilde{\psi}_{r_0}(v^L_{u_0})} = \sum\limits_{i \neq i_0} \frac{\lambda_i \lambda_{i_0}}{\lambda_{i_0} - \lambda_i} \times m_i$, where $m_i := \dim \widetilde{D}_i$. Let L be the leaf (which is a totally umbilic sphere in $H^0([0,1], \mathfrak{g})$) of $\widetilde{D}_{i_0}$ through u_0 and u_0^* the anti-podal point of u_0 in the sphere L. Similarly we can show $\mathrm{Tr}\, A^{\widetilde{F}}_{\widetilde{\psi}_{r_0}(v^L_{u_0^*})} = \sum\limits_{i \neq i_0} \dfrac{\lambda_i \lambda_{i_0}}{\lambda_{i_0} - \lambda_i} \times m_i$. Thus we have $\mathrm{Tr}\, A^{\widetilde{F}}_{\widetilde{\psi}_{r_0}(v^L_{u_0})} = \mathrm{Tr}\, A^{\widetilde{F}}_{\widetilde{\psi}_{r_0}(v^L_{u_0^*})}$. On the other hand, it follows from $\widetilde{\psi}_{r_0}(v^L_{u_0^*}) = -\widetilde{\psi}_{r_0}(v^L_{u_0})$ that $\mathrm{Tr}\, A^{\widetilde{F}}_{\widetilde{\psi}_{r_0}(v^L_{u_0})} = -\mathrm{Tr}\, A^{\widetilde{F}}_{\widetilde{\psi}_{r_0}(v^L_{u_0^*})}$. Hence we obtain

$$\mathrm{Tr}\, A^{\widetilde{F}}_{\widetilde{\psi}_{r_0}(v^L_{u_0})} = 0. \tag{3.4}$$

It follows from (i) and (ii) of Lemma 3.2 that $F := f_{r_0}(M)$ is curvature-adapted. Hence,

according to Fact 2 stated in Introduction, we have

$$\mathrm{Tr}\, A^{\widetilde F}_{\widetilde\psi_{r_0}(v^L_{u_0})} = \mathrm{Tr}\, A^F_{\psi_{r_0}(v_{x_0})}. \tag{3.5}$$

Therefore we obtain $\mathrm{Tr}\, A^F_{(\psi_{r_0}(v_{x_0}))} = 0$. This completes the proof. q.e.d.

4 The mean curvature of a proper anti-Kaehlerian Fredholm submanifold

In this section, we define the notion of a proper anti-Kaehlerian Fredholm submanifold and its mean curvature vector. Let M be an anti-Kaehlerian Fredholm submanifold in an infinite dimensional anti-Kaehlerian space V and A be the shape tensor of M. Denote by the same symbol J the complex structures of M and V. If A_v is diagonalized with respect to a J-orthonormal base for each unit normal vector v of M, then we call M a *proper anti-Kaehlerian Fredholm submanifold*. Assume that M is such a submanifold. Let v be a unit normal vector of M. If the series $\sum_{i=1}^{\infty} m_i\lambda_i$ exists, then we call it the *J-trace of A_v* and denote it by $\mathrm{Tr}_J A_v$, where $\{\lambda_i \,|\, i = 1, 2, \cdots\} = \mathrm{Spec}_J A_v \setminus \{0\}$ (λ_i's are ordered as stated in Section 2) and $m_i = \frac{1}{2}\mathrm{dimKer}(A_v - \lambda_i I)$ $(i = 1, 2, \cdots)$, where $\lambda_i I$ means $(\mathrm{Re}\,\lambda_i)I + (\mathrm{Im}\,\lambda_i)J$. Note that, if $\sharp(\mathrm{Spec}_J A_v)$ is finite, then we promise $\lambda_i = 0$ and $m_i = 0$ $(i > \sharp(\mathrm{Spec}_J A_v \setminus \{0\}))$, where $\sharp(\cdot)$ is the cardinal number of $(\cdot)$. Define a normal vector field H of M by $\langle H_x, v\rangle = \mathrm{Tr}_J A_v$ $(x \in M,\ v \in T^\perp_x M)$. We call H the *mean curvature vector of M*.

Let G/K be a symmetric space of non-compact type and $\phi : H^0([0,1], \mathfrak{g}^{\mathbf{c}}) \to G^{\mathbf{c}}$ be the parallel transport map for the complexification $G^{\mathbf{c}}$ of G and π be the natural projection of $G^{\mathbf{c}}$ onto the anti-Kaehlerian symmetric space $G^{\mathbf{c}}/K^{\mathbf{c}}$. We have the following fact, which will be used in the proof of Theorem B in the next section.

Lemma 4.1. *Let M be a curvature-adapted anti-Kaehlerian submanifold in $G^{\mathbf{c}}/K^{\mathbf{c}}$ and A (resp. $\widetilde A$) be the shape tensor of M (resp. $(\pi\circ\phi)^{-1}(M)$). Assume that, for each unit normal vector v of M and each J-eigenvalue μ of $R(v)$, $\mathrm{Ker}(A_v - \sqrt{-\mu}I) \cap \mathrm{Ker}(R(v) - \mu I) = \{0\}$ holds. Then the following statements* (i) *and* (ii) *hold:*

(i) *$(\pi\circ\phi)^{-1}(M)$ is a proper anti-Kaehlerian Fredholm submanifold.*

(ii) *For each unit normal vector v of M, $\mathrm{Tr}_J \widetilde A_{v^L} = \mathrm{Tr}_J A_v$ holds, where v^L is the horizontal lift of v to $(\pi\circ\phi)^{-1}(M)$ and $\mathrm{Tr}_J A_v$ is the J-trace of A_v.*

Proof. We can show the statement (i) in terms of Lemmas 9, 12 and 13 in [Koi3]. By imitating the proof of Theorem C in [Koi2], we can show the statement (ii), where we also use the above lemmas in [Koi3].

q.e.d.

5 Proofs of Theorems B, C and D

In this section, we first prove Theorem B. Let M be a curvature-adapted isoparametric C^ω-hypersurface in a symmetric space G/K of non-compact type. Assume that M admits no focal point of non-Euclidean type on the ideal boundary of G/K. Denote by A the shape tensor of M and R the curvature tensor of G/K. Let v be a unit normal vector field of M, which is uniquely extended to a unit normal vector field of the extrinsic complexification $M^{\mathbf{c}}(\subset G^{\mathbf{c}}/K^{\mathbf{c}})$ of M. Since M is a curvature-adapted isoparametric hypersurface admitting no focal point of non-Euclidean type on the ideal boundary $N(\infty)$, it admits a complex focal radius. Let r_0 be one of complex focal radii of M. The focal map $f_{r_0}: M^{\mathbf{c}} \to G^{\mathbf{c}}/K^{\mathbf{c}}$ for r_0 is defined by $f_{r_0}(x) := \exp^\perp(r_0 v_x)(= \gamma^{\mathbf{c}}_{v_x}(r_0))$ $(x \in M^{\mathbf{c}})$, where $r_0 v_x$ means $(\mathrm{Re} r_0)v_x + (\mathrm{Im} r_0)Jv_x$ (J : the complex structure of $G^{\mathbf{c}}/K^{\mathbf{c}}$). Let $F := f_{r_0}(M^{\mathbf{c}})$, which is an anti-Kaehlerian submanifold in $G^{\mathbf{c}}/K^{\mathbf{c}}$ (see Fig. 1). Without loss of generality, we may assume $o := eK \in M$. Denote by $\widehat{A}$ and A^F the shape tensor of $M^{\mathbf{c}}$ and F, respectively. Let ψ_t be the geodesic flow of $G^{\mathbf{c}}/K^{\mathbf{c}}$. Then we have the following fact.

Lemma 5.1. *For any $x \in M\,(\subset M^{\mathbf{c}})$, the following relation holds:*

$$\mathrm{Tr}_J A^F_{\psi_{|r_0|}(\frac{r_0}{|r_0|}v_x)} = -\frac{r_0}{|r_0|}\sum_{(\lambda,\mu)\in S^x_{r_0}} \frac{\mu + \lambda\hat{\tau}_{r_0}(\mu)}{\lambda - \hat{\tau}_{r_0}(\mu)} \times m_{\lambda,\mu},$$

where $S^x_{r_0}$ and $m_{\lambda,\mu}$ are as in the statement of Theorem B.

Proof. Let $S_x := \{(\lambda,\mu) \in \mathrm{Spec}A_{v_x} \times \mathrm{Spec}R(v_x) \,|\, \mathrm{Ker}(A_{v_x} - \lambda I) \cap \mathrm{Ker}(R(v_x) - \mu I) \neq \{0\}\}$. Since M is curvature adapted, we have $T_xM = \underset{(\lambda,\mu)\in S_x}{\oplus}(\mathrm{Ker}(A_x - \lambda I) \cap \mathrm{Ker}(R(v_x) - \mu I))$. Set $D_x := \underset{(\lambda,\mu)\in S^x_{r_0}}{\oplus}(\mathrm{Ker}(A_x - \lambda I) \cap \mathrm{Ker}(R(v_x) - \mu I))$ and $D_x^\perp$ the orthogonal complement of D_x in T_xM. The tangent space $T_x(M^{\mathbf{c}})$ is identified with the complexification $(T_xM)^{\mathbf{c}}$. Under this identification, the shape operator $\widehat{A}_{v_x}$ is identified with the complexification $A^{\mathbf{c}}_x$ of A_x. Let $X \in \mathrm{Ker}(A_x - \lambda I)^{\mathbf{c}} \cap \mathrm{Ker}(R(v_x) - \mu I)^{\mathbf{c}}$ $((\lambda,\mu) \in S^x_{r_0})$ and Y be the Jacobi field along $\gamma_{r_0 v_x}$ with $Y(0) = X$ and $Y'(0) = -\hat{A}_{r_0 v_x}X\,(= -r_0\lambda X = -\lambda\,((\mathrm{Re}r_0)X + (\mathrm{Im}r_0)JX))$, where $\gamma_{r_0 v_x}$ is the geodesic in $G^{\mathbf{c}}/K^{\mathbf{c}}$ with $\dot\gamma_{r_0 v_x}(0) = r_0 v_x(= (\mathrm{Re}r_0)v_x + (\mathrm{Im}r_0)Jv_x)$. This Jacobi field Y is described as

$$Y(s) = \left(\cos(\mathbf{i}sr_0\sqrt{-\mu}) - \frac{\lambda\sin(\mathbf{i}sr_0\sqrt{-\mu})}{\mathbf{i}\sqrt{-\mu}}\right)P_{\gamma_{r_0 v_x}|_{[0,s]}}(X).$$

Since $Y(1) = f_{r_0*}X$, we have

$$f_{r_0*}X = \left(\cos(\mathbf{i}r_0\sqrt{-\mu}) - \frac{\lambda\sin(\mathbf{i}r_0\sqrt{-\mu})}{\mathbf{i}\sqrt{-\mu}}\right)P_{\gamma_{r_0 v_x}}(X) \tag{5.1}$$

which is not equal to 0 because of $(\lambda,\mu) \in S^x_{r_0}$. This relation implies that $T_{f_{r_0}(x)}F = P_{\gamma_{r_0 v_x}}(D^{\mathbf{c}}_x)$. On the other hand, we have

$$\begin{aligned}\widetilde{\nabla}_{f_{r_0*}X}\psi_{|r_0|}(\frac{r_0}{|r_0|}v_x) &= \frac{1}{|r_0|}Y'(1)\\ &= -\frac{r_0}{|r_0|}\left(\mathbf{i}\sqrt{-\mu}\sin(\mathbf{i}r_0\sqrt{-\mu}) + \lambda\cos(\mathbf{i}r_0\sqrt{-\mu})\right)P_{\gamma_{r_0 v_x}}(X).\end{aligned} \tag{5.2}$$

From (5.1) and (5.2), we have

$$A^F_{\psi_{|r_0|}(\frac{r_0}{|r_0|}v_x)} f_{r_0*}X = \frac{-\frac{r_0}{|r_0|}(\mu + \lambda\hat{\tau}_{r_0}(\mu))}{\lambda - \hat{\tau}_{r_0}(\mu)} f_{r_0*}X. \tag{5.3}$$

The desired relation follows from this relation. q.e.d.

Set $\kappa(\lambda,\mu) := \frac{-\frac{r_0}{|r_0|}(\mu+\lambda\hat{\tau}_{r_0}(\mu))}{\lambda-\hat{\tau}_{r_0}(\mu)}$ $((\lambda,\mu)\in S^x_{r_0})$. Next we prepare the following lemma.

Lemma 5.2. *Let* $(\lambda_1,\mu_1)\in S^x_{r_0}$. *Then we have*

(i) $(\exp_{G^{\mathbf{c}}} r_0 v_x)_*^{-1}\psi_{|r_0|}(\frac{r_0}{|r_0|}v_x) = \frac{r_0}{|r_0|}v_x$, *where* $\exp_{G^{\mathbf{c}}}$ *is the exponential map of* $G^{\mathbf{c}}$,

(ii) $$(\exp_{G^{\mathbf{c}}} r_0 v_x)_*^{-1}\left(\mathrm{Ker}(A^F_{\psi_{|r_0|}(\frac{r_0}{|r_0|}v_x)} - \kappa(\lambda_1,\mu_1)I)\right) = \bigoplus_{(\lambda,\mu)\in S^x_{r_0}(\lambda_1,\mu_1)} (\mathrm{Ker}(A_{v_x}-\lambda I)^{\mathbf{c}}\cap \mathrm{Ker}(R(v_x)-\mu I)^{\mathbf{c}}),$$

where $S^x_{r_0}(\lambda_1,\mu_1) = \{(\lambda,\mu)\in S^x_{r_0}\,|\,\kappa(\lambda,\mu)=\kappa(\lambda_1,\mu_1)\}$,

(iii) *if* $\lambda_1 \neq \pm\sqrt{-\mu_1}$, *then* $\kappa(\lambda_1,\mu_1)\neq \pm\frac{r_0}{|r_0|}\sqrt{-\mu_1}$.

Proof. The relation of (i) is trivial. Let $(\lambda,\mu)\in S^x_{r_0}(\lambda_1,\mu_1)$. The restriction $f_{r_0*}|_{\mathrm{Ker}(A_{v_x}-\lambda I)^{\mathbf{c}}\cap\mathrm{Ker}(R(v_x)-\mu I)^{\mathbf{c}}}$ of f_{r_0*} is equal to $P_{\gamma_{r_0v_x}}|_{\mathrm{Ker}(A_{v_x}-\lambda I)^{\mathbf{c}}\cap\mathrm{Ker}(R(v_x)-\mu I)^{\mathbf{c}}}$ up to constant multiple by (5.1). Also, we have $P_{\gamma_{r_0v_x}} = (\exp_{G^{\mathbf{c}}} r_0v_x)_*$. These facts together with (5.3) deduce

$$\begin{aligned}&(\exp_{G^{\mathbf{c}}} r_0v_x)_*(\mathrm{Ker}(A_{v_x}-\lambda I)^{\mathbf{c}}\cap\mathrm{Ker}(R(v_x)-\mu I)^{\mathbf{c}})\\ &= f_{r_0*}(\mathrm{Ker}(A_{v_x}-\lambda I)^{\mathbf{c}}\cap\mathrm{Ker}(R(v_x)-\mu I)^{\mathbf{c}})\\ &\subset \mathrm{Ker}\left(A^F_{\psi_{|r_0|}(\frac{r_0}{|r_0|}v_x)} - \kappa(\lambda_1,\mu_1)I\right).\end{aligned}$$

From this fact, the relation of (ii) follows. Now we shall show the statement (iii). Let $r_0 = a_0 + b_0\sqrt{-1}$ $(a_0,b_0\in\mathbf{R})$. Suppose that $\kappa(\lambda_1,\mu_1) = \pm\frac{r_0}{|r_0|}\sqrt{-\mu_1}$. By squaring both sides of this relation, we have

$$\left(\hat{\tau}_{r_0}(\mu_1)^2+\mu_1\right)\left(\lambda_1^2+\mu_1\right)=0.$$

Hence we have $\lambda_1 = \pm\sqrt{-\mu_1}$. Thus the statement (iii) is shown. q.e.d.

Denote by $\hat{R}$ the curvature tensor of $G^{\mathbf{c}}/K^{\mathbf{c}}$. By using these lemmas, we prove Theorem B. According to Lemma 5.1, we have only to show $\mathrm{Tr}_J A^F_{\psi_{|r_0|}(\frac{r_0}{|r_0|}v_x)} = 0$ $(x\in M)$. In the case where M is homogeneous, we can show this relation by imitating the process of the proof of Corollary 1.1 of [HL].

Simple proof of Theorem B in rank one case. We have only to show $\mathrm{Tr}_J A^F_{\psi_{|r_0|}(\frac{r_0}{|r_0|}v_x)} = 0$. Assume that G/K is of rank one. Define a complex linear function $\Phi : T^\perp_{f_{r_0}(x)}F \to \mathbf{C}$ by $\Phi(w) = \mathrm{Tr}_J A^F_w$ $(w\in T^\perp_{f_{r_0}(x)}F)$. Since M is curvature-adapted, we have $T_xM =$

$\underset{(\lambda,\mu)\in S_x}{\oplus}(\mathrm{Ker}(A_{v_x}-\lambda I)\cap\mathrm{Ker}(R(v_x)-\mu I))$. Set

$$\hat S^y_{r_0}:=\{(\lambda,\mu)\in(\mathrm{Spec}_J\hat A_{v_y})\times(\mathrm{Spec}_J\hat R(v_y))\,|\,\mathrm{Ker}(\hat A_{v_y}-\lambda I)\cap\mathrm{Ker}(\hat R(v_y)-\mu I)\neq\{0\}\\ \&\,\lambda\neq\hat f_{r_0}(\mu)\}$$

$(y\in M^{\mathbf c})$. Define a distribution $\hat D$ on $M^{\mathbf c}$ by $\hat D_y:=\underset{(\lambda,\mu)\in\hat S^y_{r_0}}{\oplus}\left(\mathrm{Ker}(\hat A_{v_y}-\lambda I)\cap\mathrm{Ker}(\hat R(v_y)-\mu I)\right)$ $(y\in M^{\mathbf c})$ and $\hat D^\perp$ the orthogonal complementary distribution of $\hat D$ in $T(M^{\mathbf c})$. Also, define a distribution D on M by $D_x:=\underset{(\lambda,\mu)\in\hat S^x_{r_0}}{\oplus}(\mathrm{Ker}(A_x-\lambda I)\cap\mathrm{Ker}(R(v_x)-\mu I))$ $(x\in M)$ and $D^\perp$ the orthogonal complementary distribution of D in TM. Under the identification of $T_x(M^{\mathbf c})$ with $(T_xM)^{\mathbf c}$, $\hat D_x$ is identified with the complexification $(D_x)^{\mathbf c}$ of D_x. The focal map f_{r_0} is a submersoin of $M^{\mathbf c}$ onto F and the fibres of f_{r_0} are integral manifolds of $\hat D^\perp$. Let L be the integral manifold of $\hat D^\perp$ through x and set $L_{\mathbf R}:=L\cap M$. It is shown that L is the extrinsic complexification of $L_{\mathbf R}$. Set $Q:=\{\psi_{|r_0|}(\frac{r_0}{|r_0|}v_x)\,|\,x\in L\}$ and $Q_{\mathbf R}:=\{\psi_{|r_0|}(\frac{r_0}{|r_0|}v_x)\,|\,x\in L_{\mathbf R}\}$. It is shown that Q is the extrinsic complexification of $Q_{\mathbf R}$ and that Q is a complex hypersurface without geodesic point in $T^\perp_{f_{r_0}(x)}F$, that is, it is not contained in any complex affine hyperplane of $T^\perp_{f_{r_0}(x)}F$. According to Lemma 5.1, we have

$$\Phi(\psi_{|r_0|}(\frac{r_0}{|r_0|}v_y))=-\frac{r_0}{|r_0|}\sum_{(\lambda,\mu)\in S^y_{r_0}}\frac{\mu+\lambda\hat\tau_{r_0}(\mu)}{\lambda-\hat\tau_{r_0}(\mu)}\times m_{\lambda,\mu}.$$

Let $(\widetilde\lambda,\widetilde\mu)$ be a pair of continuous functions on $L_{\mathbf R}$ such that $(\widetilde\lambda(y),\widetilde\mu(y))\in S^y_{r_0}$ for any $y\in L$. Since G/K is of rank one, $\widetilde\mu$ is constant on $L_{\mathbf R}$. The complex focal radius having $\mathrm{Ker}(A_y-\widetilde\lambda(y)\,I)\cap\mathrm{Ker}(R(v_y)-\widetilde\mu(y)\,I)$ as a part of the focal space is the complex number z_0 satisfying $\mathrm{Ker}(D^{co}_{z_0v_y}-z_0D^{si}_{z_0v_y}\circ A^{\mathbf c}_y)|_{\mathrm{Ker}(A_y-\widetilde\lambda(y)\,I)\cap\mathrm{Ker}(R(v_y)-\widetilde\mu(y)\,I)}\neq\{0\}$, that is, it is equal to $\frac{1}{\sqrt{\widetilde\mu(y)}}\arctan\frac{\sqrt{\widetilde\mu(y)}}{\widetilde\lambda(y)}$, which is independent of the choice of $y\in L_{\mathbf R}$ by the isoparametricness (hence complex equifocality) of M. Hence $\widetilde\lambda$ is constant on $L_{\mathbf R}$. Therefore Φ is constant along $Q_{\mathbf R}$. Since Φ is of class C^ω and $Q_{\mathbf R}$ is a half-dimensional totally real submanifold in Q, Φ is constant along Q. Furthermore, this fact together with the linearity of Φ imply $\Phi\equiv0$. In particular, we have $\mathrm{Tr}\,A^F_{\psi_{r_0}(v_x)}=0$. q.e.d.

Proof of Theorem B (general case). According to Lemma 5.1, we have only to show $\mathrm{Tr}_JA^F_{\psi_{|r_0|}(\frac{r_0}{|r_0|}v_{x_0})}=0$ $(x_0\in M)$. We shall show this relation by investigating the focal submanifold of $(\pi\circ\phi)^{-1}(M^{\mathbf c})$, where ϕ $(:H^0([0,1],\mathfrak g^{\mathbf c})\to G^{\mathbf c})$ is the parallel transport map for $G^{\mathbf c}$ and π is the natural projection of $G^{\mathbf c}$ onto $G^{\mathbf c}/K^{\mathbf c}$. Let $\widetilde{M^{\mathbf c}}$ be the complete extension of $(\pi\circ\phi)^{-1}(M^{\mathbf c})$. Let v^L be the horizontal lift of v to $\widetilde{M^{\mathbf c}}$. Since $\pi\circ\phi$ is an anti-Kaehlerian submersion, the complex focal radii of $M^{\mathbf c}$ (hence M) are those of $\widetilde{M^{\mathbf c}}$. Let r_0 be a complex focal radius of M (hence $\widetilde{M^{\mathbf c}}$). The focal map $\widetilde f_{r_0}$ for r_0 is defined by $\widetilde f_{r_0}(x)=x+r_0v^L_x$ $(x\in\widetilde{M^{\mathbf c}})$. Set $\widetilde F:=\widetilde f_{r_0}(\widetilde{M^{\mathbf c}})$. Denote by $\widetilde A$ (resp. $A^{\widetilde F}$) the shape tensor of $\widetilde{M^{\mathbf c}}$ (resp. $\widetilde F$). Let $\mathrm{Spec}_J\widetilde A_{v^L_{\hat 0}}\setminus\{0\}=\{\lambda_i\,|\,i=1,2,\cdots\}$ ("$|\lambda_i|>|\lambda_{i+1}|$" or "$|\lambda_i|=$

$|\lambda_{i+1}|$ & $\mathrm{Re}\lambda_i > \mathrm{Re}\lambda_{i+1}$" or "$|\lambda_i| = |\lambda_{i+1}|$ & $\mathrm{Re}\lambda_i = \mathrm{Re}\lambda_{i+1}$ & $\mathrm{Im}\lambda_i = -\mathrm{Im}\lambda_{i+1} > 0$"). The set of all complex focal radii of $M^{\mathbf{c}}$ (hence M) is equl to $\{\frac{1}{\lambda_i}\,|\,i = 1, 2, \cdots\}$. We have $r_0 = \frac{1}{\lambda_{i_0}}$ for some i_0. Define a distribution $\widetilde{D}_i$ $(i = 0, 1, 2, \cdots)$ on $\widetilde{M^{\mathbf{c}}}$ by $(\widetilde{D}_0)_u := \mathrm{Ker}\widetilde{A}_{\widetilde{v}^L_u}$ and $(\widetilde{D}_i)_u := \mathrm{Ker}(\widetilde{A}_{\widetilde{v}^L_u} - \lambda_i I)$ $(i = 1, 2, \cdots)$, where $u \in \widetilde{M^{\mathbf{c}}}$. Since M is a curvature-adapted isoparametric submanifold admitting no focal point of non-Euclidean type on $N(\infty)$, $\widetilde{M^{\mathbf{c}}}$ is proper anti-Kaehlerian isoparametric by Fact 5. Therefore, we have $T\widetilde{M^{\mathbf{c}}} = \overline{\widetilde{D}_0 \oplus (\bigoplus_i \widetilde{D}_i)}$ and $\mathrm{Spec}_J \widetilde{A}_{\widetilde{v}^L_u}$ is independent of the choice of $u \in \widetilde{M^{\mathbf{c}}}$. Take $u_0 \in \widetilde{M^{\mathbf{c}}}$ with $(\pi \circ \phi)(u_0) = x_0$. Let $X_i \in (\widetilde{D}_i)_{u_0}$ $(i \neq i_0)$ and $X_0 \in (\widetilde{D}_0)_{u_0}$. Then we have $\widetilde{f}_{r_0*}X_i = (1 - r_0\lambda_i)X_i$ and $\widetilde{f}_{r_0*}X_0 = X_0$. Hence we have $T_{\widetilde{f}_{r_0}(u_0)}\widetilde{F} = (\widetilde{D}_0)_{u_0} \oplus (\bigoplus_{i \neq i_0} (\widetilde{D}_i)_{u_0})$ and $\mathrm{Ker}(\widetilde{f}_{r_0})_{*u_0} = (\widetilde{D}_{i_0})_{u_0}$, which implies that $\widetilde{D}_{i_0}$ is integrable. On the other hand, we have $A^{\widetilde{F}}_{\widetilde{\psi}_{|r_0|}(\frac{r_0}{|r_0|}v^L_{u_0})}\widetilde{f}_{r_0*}X_i = \dfrac{\lambda_i r_0}{|r_0|}X_i$ and $A^{\widetilde{F}}_{\widetilde{\psi}_{|r_0|}(\frac{r_0}{|r_0|}v^L_{u_0})}\widetilde{f}_{r_0*}X_0 = 0$, where $\widetilde{\psi}$ is the geodesic flow of $H^0([0,1], \mathfrak{g}^{\mathbf{c}})$. Therefore, we obtain $A^{\widetilde{F}}_{\widetilde{\psi}_{|r_0|}(\frac{r_0}{|r_0|}v^L_{u_0})}\widetilde{f}_{r_0*}X_i = \dfrac{\lambda_i|\lambda_{i_0}|}{\lambda_{i_0} - \lambda_i}\widetilde{f}_{r_0*}X_i$. Hence we have $\mathrm{Tr}_J A^{\widetilde{F}}_{\widetilde{\psi}_{|r_0|}(\frac{r_0}{|r_0|}v^L_{u_0})} = \sum_{i \neq i_0} \frac{\lambda_i|\lambda_{i_0}|}{\lambda_{i_0} - \lambda_i} \times m_i$, where $m_i := \frac{1}{2}\dim \widetilde{D}_i$. According to Theorem 2 of [Koi3], each leaf of $\widetilde{D}_{i_0}$ is a complex sphere. Let L be the leaf of $\widetilde{D}_{i_0}$ through u_0 and u_0^* be the anti-podal point of u_0 in the complex sphere L. Similarly we can show $\mathrm{Tr}_J A^{\widetilde{F}}_{\widetilde{\psi}_{|r_0|}(\frac{r_0}{|r_0|}(\widetilde{v}^L)_{u_0^*})} = \displaystyle\sum_{i \neq i_0} \frac{\lambda_i|\lambda_{i_0}|}{\lambda_{i_0} - \lambda_i} \times m_i$. Thus we have $\mathrm{Tr}_J A^{\widetilde{F}}_{\widetilde{\psi}_{|r_0|}(\frac{r_0}{|r_0|}v^L_{u_0})} = \mathrm{Tr}_J A^{\widetilde{F}}_{\widetilde{\psi}_{|r_0|}(\frac{r_0}{|r_0|}(\widetilde{v}^L)_{u_0^*})}$. On the other hand, it follows from $\widetilde{\psi}_{|r_0|}(\frac{r_0}{|r_0|}(\widetilde{v}^L)_{u_0^*}) = -\widetilde{\psi}_{|r_0|}(\frac{r_0}{|r_0|}v^L_{u_0})$ that $\mathrm{Tr}_J A^{\widetilde{F}}_{\widetilde{\psi}_{|r_0|}(\frac{r_0}{|r_0|}v^L_{u_0})} = -\mathrm{Tr}_J A^{\widetilde{F}}_{\widetilde{\psi}_{|r_0|}(\frac{r_0}{|r_0|}(\widetilde{v}^L)_{u_0^*})}$. Hence we obtain

$$\mathrm{Tr}_J A^{\widetilde{F}}_{\widetilde{\psi}_{|r_0|}(\frac{r_0}{|r_0|}v^L_{u_0})} = 0. \tag{5.4}$$

It follows from (i) and (ii) of Lemma 5.2 that $F := f_{r_0}(M^{\mathbf{c}})$ is a curvature adapted anti-Kaehlerian submanifold. Also, it follows from (iv) of Remark 1.2, (5.3), (i) and (iii) of Lemma 5.2 that, for each unit normal vector w of F and each $\mu \in \mathrm{Spec}_J R(w) \setminus \{0\}$, $\mathrm{Ker}(A^F_w \pm \sqrt{-\mu}I) \cap \mathrm{Ker}(R(w) - \mu I) = \{0\}$ holds. Therefore, it follows from Lemma 4.1 that $\widetilde{F}$ is a proper anti-Kaehlerian Fredholm submanifold and, for each unit normal vector w of F, we have $\mathrm{Tr}_J A^{\widetilde{F}}_{w^L} = \mathrm{Tr}_J A^F_w$. It is clear that $\widetilde{\psi}_{|r_0|}(\frac{r_0}{|r_0|}v^L_{u_0})$ is the horizontal lift of $\psi_{|r_0|}(\frac{r_0}{|r_0|}v_{x_0})$ to $\widetilde{f}_{r_0}(u_0)$. Hence we have

$$\mathrm{Tr}_J A^F_{\psi_{|r_0|}(\frac{r_0}{|r_0|}v_{x_0})} = \mathrm{Tr}_J A^{\widetilde{F}}_{\widetilde{\psi}_{|r_0|}(\frac{r_0}{|r_0|}v^L_{u_0})}, \tag{5.5}$$

, From (5.4) and (5.5), we have $\mathrm{Tr}_J A^F_{\psi_{|r_0|}(\frac{r_0}{|r_0|}v_{x_0})} = 0$. This completes the proof. q.e.d.

Next we prove Theorem C in terms of Theorem B.

Proof of Theorem C. Let r_0 be a complex focal radius of M with $\mathrm{Re}r_0 = \max_r \mathrm{Re}r$, where r runs over the set of all complex focal radii of M. Let $(\lambda, \mu) \in S^x_{r_0} \setminus \{(0,0)\}$ and r a

complex focal radius including $\mathrm{Ker}(A_v-\lambda I)\cap\mathrm{Ker}(R(v)-\mu I)$ as the focal space, that is, $\lambda=\hat\tau_r(\mu)$ (see (ii) of Remark 1.2). Set $c_{\lambda,\mu}:=-\frac{\mu+\lambda\hat\tau_{r_0}(\mu)}{\lambda-\hat\tau_{r_0}(\mu)}$. We shall show $\mathrm{Re}\,c_{\lambda,\mu}\le 0$. We divide into the following three cases:

$$\text{(i) } \mu=0 \quad \text{(ii) } 0<\sqrt{-\mu}<|\lambda| \quad \text{(iii) } |\lambda|<\sqrt{-\mu}.$$

First we consider the case (i). Then we have $c_{\lambda,\mu}=\frac{\lambda}{1-\lambda r_0}$. Also, we can show $\lambda=\frac{1}{r}$. Hence we have

$$c_{\lambda,\mu}=\frac{1}{r-r_0}. \tag{5.6}$$

Furthermore, we have $\mathrm{Re}\,c_{\lambda,\mu}\le 0$ from the choice of r_0. Next we consider the case (ii). Since $\lambda=\hat\tau_r(\mu)$ and λ is a real number with $|\lambda|>\sqrt{-\mu}$, we can show $\lambda=\hat\tau_{\mathrm{Re}\,r}(\mu)(=\frac{\sqrt{-\mu}}{\tanh(\sqrt{-\mu}\mathrm{Re}\,r)})$ and $r\equiv\mathrm{Re}\,r \pmod{\frac{\pi\mathbf{i}}{\sqrt{-\mu}}}$. Hence we have $c_{\lambda,\mu}=\hat\tau_{(r_0-\mathrm{Re}\,r)}(\mu)$, where we note that $\mathrm{Re}r\not\equiv r_0 \pmod{\frac{\pi\mathbf{i}}{\sqrt{-\mu}}}$ because of $(\lambda,\mu)\in S^x_{r_0}$. Therefore, we obtain

$$\mathrm{Re}\,c_{\lambda,\mu}=\frac{\sqrt{-\mu}\left(1+\tan^2(\sqrt{-\mu}\mathrm{Im}r_0)\right)\tanh(\sqrt{-\mu}(\mathrm{Re}r-\mathrm{Re}r_0))}{\tanh^2(\sqrt{-\mu}(\mathrm{Re}r-\mathrm{Re}r_0))+\tan^2(\sqrt{-\mu}\mathrm{Im}r_0)}\le 0 \tag{5.7}$$

because of $\mathrm{Re}r\le\mathrm{Re}r_0$. Next we consider the case (iii). Since $\lambda=\hat\tau_r(\mu)$ and λ is a real number with $|\lambda|<\sqrt{-\mu}$, we can show $\lambda=\hat\tau_{(\mathrm{Re}\,r+\frac{\pi\mathbf{i}}{2\sqrt{-\mu}})}(\mu)(=\sqrt{-\mu}\tanh(\sqrt{-\mu}\mathrm{Re}\,r))$ and $r\equiv\mathrm{Re}\,r+\frac{\pi\mathbf{i}}{2\sqrt{-\mu}} \pmod{\frac{\pi\mathbf{i}}{\sqrt{-\mu}}}$. Hence we have $c_{\lambda,\mu}=\hat\tau_{(r_0-\mathrm{Re}r+\frac{\pi\mathbf{i}}{2\sqrt{-\mu}})}(\mu)$. Therefore, we obtain

$$\mathrm{Re}c_{\lambda,\mu}=\frac{\sqrt{-\mu}\left(1+\tan^2(\sqrt{-\mu}\mathrm{Im}r_0)\right)\tanh(\sqrt{-\mu}(\mathrm{Re}r-\mathrm{Re}r_0))}{1+\tanh^2(\sqrt{-\mu}(\mathrm{Re}r-\mathrm{Re}r_0))\tan^2(\sqrt{-\mu}\mathrm{Im}r_0)}\le 0. \tag{5.8}$$

Thus $\mathrm{Re}c_{\lambda,\mu}\le 0$ is shown in general. Hence, from the identity in Theorem B, $\mathrm{Re}c_{\lambda,\mu}=0$ ($(\lambda,\mu)\in S^x_{r_0}$) follows, where we note that $c_{0,0}=0$. In case of (i), it follows from (5.6) that $\mathrm{Re}\left(\frac{1}{r-r_0}\right)=0$. Hence we have $\mathrm{Re}\,r=\mathrm{Re}\,r_0(<\infty)$ or $r=\infty$. If $\mathrm{Re}\,r=\mathrm{Re}\,r_0(<\infty)$, then we have $\lambda=\frac{1}{r}=\frac{1}{\mathrm{Re}\,r_0}=\hat\tau_{\mathrm{Re}\,r_0}(0)$ (which does not happen if r_0 is real because of $(\lambda,0)\in S^x_{r_0}$). Also, if $r=\infty$, then we have $\lambda=0$. Thus we have

$$\mathrm{Spec}(A_x|_{D_0})\subset\{\frac{1}{\mathrm{Re}\,r_0},0\}. \tag{5.9}$$

In case of (ii), it follows from (5.7) that $\mathrm{Re}r=\mathrm{Re}r_0$. Hence we have $\lambda=\hat\tau_{\mathrm{Re}\,r_0}(\mu)$ (which does not happen if $r_0\equiv\mathrm{Re}\,r_0 \pmod{\frac{\pi\mathbf{i}}{\sqrt{-\mu}}}$ because of $(\lambda,\mu)\in S^x_{r_0}$). In case of (iii), it follows from (5.8) that $\mathrm{Re}r=\mathrm{Re}r_0$. Hence we have $\lambda=\hat\tau_{(\mathrm{Re}\,r_0+\frac{\pi\mathbf{i}}{2\sqrt{-\mu}})}(\mu)$ (which does not happen if $r_0\equiv\mathrm{Re}\,r_0+\frac{\pi\mathbf{i}}{2\sqrt{-\mu}} \pmod{\frac{\pi\mathbf{i}}{\sqrt{-\mu}}}$ because of $(\lambda,\mu)\in S^x_{r_0}$). Hence we have

$$\mathrm{Spec}(A_x|_{D_\mu})\subset\{\frac{\sqrt{-\mu}}{\tanh(\sqrt{-\mu}\mathrm{Re}r_0)},\sqrt{-\mu}\tanh(\sqrt{-\mu}\mathrm{Re}r_0)\}. \tag{5.10}$$

Thus the statement (i) is shown. Furthermore, from

$$\mathrm{Spec}R(v_x)=\begin{cases}\{-\beta(g_*^{-1}v_x)^2\,|\,\beta\in\triangle_+|_{\mathbf{R}v_x}\}\cup\{0\} & (\mathrm{rank}(G/K)\ge 2)\\ \{-\beta(g_*^{-1}v_x)^2\,|\,\beta\in\triangle_+|_{\mathbf{R}v_x}\} & (\mathrm{rank}(G/K)=1)\end{cases}$$

and

$$\dim \operatorname{Ker} R(v_x) \begin{cases} \geq 2 & (\operatorname{rank}(G/K) \geq 3) \\ = 1 & (\operatorname{rank}(G/K) = 2), \end{cases}$$

the statement (ii) follows. q.e.d.

Next we prove Theorem D in terms of Theorem C and its proof.

Proof of Theorem D. For each $\beta \in \triangle_+|_{\mathbf{R}v_x}$, we set $D_\beta := \operatorname{Ker}(R(v_x) + \beta(g_*^{-1}v_x)^2 \mathrm{id})$. According to the proof of Theorem C, the real parts of complex focal radii of M coincide with one another. Denote by s_0 this real part. Furthermore, from (5.9) and (5.10), we have

$$\operatorname{Spec}(A_x|_{D_0}) \subset \{\frac{1}{s_0}, 0\}$$

and

$$\operatorname{Spec}(A_x|_{D_\beta}) \subset \{\frac{\beta(g_*^{-1}v_x)}{\tanh(\beta(g_*^{-1}v_x)s_0)},\ \beta(g_*^{-1}v_x)\tanh(\beta(g_*^{-1}v_x)s_0)\} \quad (\beta \in \triangle_+|_{\mathbf{R}v_x}).$$

Set $D_0^V := \operatorname{Ker}\left(A_x|_{D_0} - \frac{1}{s_0}\mathrm{id}\right)$, $D_0^H := \operatorname{Ker} A_x|_{D_0}$,

$$D_\beta^V := \operatorname{Ker}\left(A_x|_{D_\beta} - \frac{\beta(g_*^{-1}v_x)}{\tanh(\beta(g_*^{-1}v_x)s_0)}\mathrm{id}\right)$$

and

$$D_\beta^H := \operatorname{Ker}\left(A_x|_{D_\beta} - \beta(g_*^{-1}v_x)\tanh(\beta(g_*^{-1}v_x)s_0)\mathrm{id}\right).$$

According to (ii) of Remark 1.2, s_0 is a (real) focal radius of M whose focal space includes D_0^V and D_β^V's ($\beta \in \triangle_+|_{\mathbf{R}v_x}$). Also the focal space does not include D_0^H and D_β^H's ($\beta \in \triangle_+|_{\mathbf{R}v_x}$). Hence the focal space coincides with $D_0^V \oplus \left(\underset{\beta \in \triangle_+|_{\mathbf{R}v_x}}{\oplus} D_\beta^V\right)$. Let η_{sv} ($s \in \mathbb{R}$) be the end-point map for sv. Set $M_s := \eta_{sv}(M)$. In particular, we set $F := M_{s_0}$, which is the only focal submanifold of M. Define a unit normal vector field v^s of M_s ($0 \leq s < s_0$) by $v^s_{\eta_{sv}(x)} = \gamma'_{v_x}(s)$ ($x \in M$). Denote by A^s ($0 \leq s < s_0$) the shape operator of M_s (for v^s) and A^F the shape tensor of F. Set $(D_0^V)^s := (\eta_{sv})_*(D_0^V)$ ($0 \leq s < s_0$) and $(D_\beta^V)^s := (\eta_{sv})_*(D_\beta^V)$ ($0 \leq s < s_0$, $\beta \in \triangle_+|_{\mathbb{R}v_x}$). Also, set $(D_0^H)^s := (\eta_{sv})_*(D_0^H)$ ($s \in \mathbb{R}$) and $(D_\beta^H)^s := (\eta_{sv})_*(D_\beta^H)$ ($s \in \mathbb{R}$, $\beta \in \triangle_+|_{\mathbb{R}v_x}$). Easily we have

$$(5.11) \qquad T_{\eta_{s_0v}(x)}F = (D_0^H)^{s_0}_{\eta_{s_0v}(x)} \oplus \left(\underset{\beta \in \triangle_+|_{\mathbf{R}v_x}}{\oplus} (D_\beta^H)^{s_0}_{\eta_{s_0v}(x)}\right).$$

Also, we can show

$$A^s_{\eta_{sv}(x)}|_{(D_0^H)^s_{\eta_{sv}(x)}} = 0 \quad (0 \leq s < s_0)$$

and

$$A^s_{\eta_{sv}(x)}|_{(D_\beta^H)^s_{\eta_{sv}(x)}} = \beta(g_*^{-1}v_x)\tanh(\beta(g_*^{-1}v_x)(s_0 - s))\,\mathrm{id} \quad (0 \leq s < s_0).$$

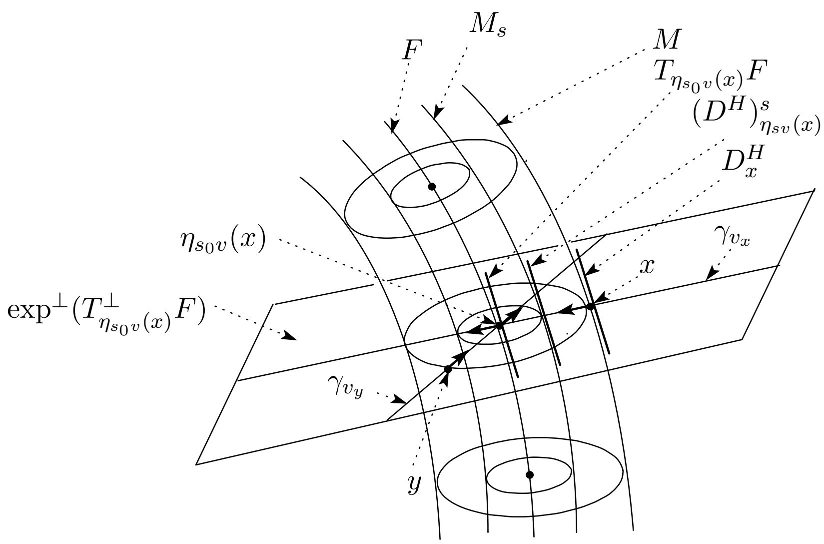
Hence we have

$$A^F_{\psi_{s_0}(v_x)}|_{(D_0^H)^{s_0}_{\eta_{s_0}v(x)}} = 0$$

and

$$A^F_{\psi_{s_0}(v_x)}|_{(D_\beta^H)^{s_0}_{\eta_{s_0}v(x)}} = \left(\lim_{s\to s_0-0}\beta(g_*^{-1}v_x)\tanh(\beta(g_*^{-1}v_x)(s_0-s))\right)\mathrm{id} = 0.$$

From these relations and (5.11), we obtain $A^F_{\psi_{s_0}(v_x)} = 0$. Since this relation holds for any $x \in M$, we have $A^F = 0$, that is, F is totally geodesic. Also, it is clear that M is the tube of radius s_0 over F. This completes the proof. q.e.d.



$$D^H_x := (D_0^H)_x \oplus \left(\underset{\beta\in\triangle_+|_{\mathbf{R}v_x}}{\oplus}(D^H_\beta)_x\right)$$
$$(D^H)^s_{\eta_{sv}(x)} := (D_0^H)^s_{\eta_{sv}(x)} \oplus \left(\underset{\beta\in\triangle_+|_{\mathbf{R}v_x}}{\oplus}(D^H_\beta)^s_{\eta_{sv}(x)}\right)$$

Fig. 3.

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